OPTIMIZED SCHWARZ METHODS FOR DOMAINS WITH CYLINDRICAL INTERFACES

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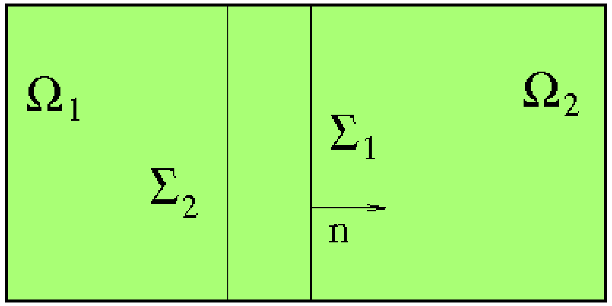
SUMMARY

- Introduction and motivations
- The diffusion-reaction problem with cylindrical interfaces
- Optimizations
- Numerical results
- Extension to the fluid-structure interaction problem
- Conclusions

INTRODUCTION



Start from an elliptic problem

$$\begin{cases} \mathcal{L}\, u = f & \boldsymbol{x} \in \Omega \\ u = 0 & \boldsymbol{x} \in \partial\Omega \end{cases}$$

Given 2 linear operators $S_1 \neq S_2$, we notice that it is equivalent to

$$\begin{cases} \mathcal{L}\, u_1 = f & \boldsymbol{x} \in \Omega_1 \\ u_1 = 0 & \boldsymbol{x} \in \partial\Omega_1 \setminus \Sigma_1 \\ (S_1 + \partial_n)u_1 = (S_1 + \partial_n)u_2 & \boldsymbol{x} \in \Sigma_1 \\ (S_2 + \partial_n)u_2 = (S_2 + \partial_n)u_1 & \boldsymbol{x} \in \Sigma_2 \\ u_2 = 0 & \boldsymbol{x} \in \partial\Omega_2 \setminus \Sigma \\ \mathcal{L}\, u_2 = f & \boldsymbol{x} \in \Omega_2 \end{cases}$$

For its solution we consider a block-Gauss-Seidel algorithm $\longrightarrow$
Generalized Schwarz method (Lions, 1990; Ciarton, Nataf, Rogier, 1991; Gander, 2006)
(for $S_1 \to \infty$ and $S_2 \to \infty$ we have the Classical Schwarz method)

Optimized Schwarz methods are obtained by looking for $S_1, S_2 \in \mathcal{C} \subset \mathcal{L}(H^{1/2}, H^{-1/2})$ which guarantee the best convergence factor in the subset $\mathcal{C}$ (Japhet, 1998)

STATE OF THE ART

Optimized Schwarz methods applied to a great variety of problems

- Advection-reaction-diffusion problems Japhet et al, FGCOS 2001; Gander, SINUM 2006;

- Helmholtz equation Gander et al, SISC 2002; Magoules et al, CMAME 2004;

- Coupling of heterogeneous media Gander and Halpern, SINUM 2007; Maday and Magoules, CMAME 2007;

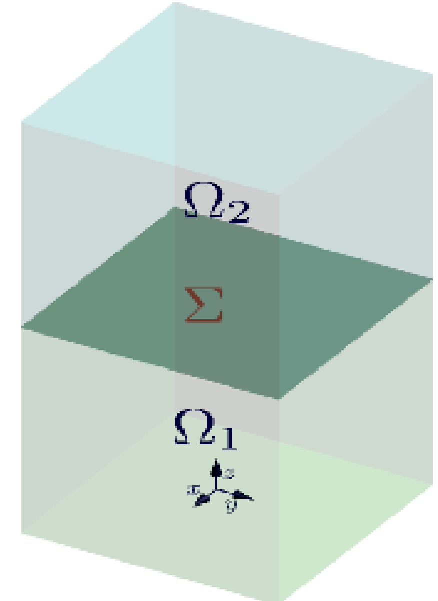
- Shallow water equations Quaddouria et al, ANM 2008;

- Maxwell's equations Dolean et al, SISC 2009;

- The scattering problem Stupfel, JCP 2010;

- The fluid-structure interaction problem Gerardo-Giorda, Nobile, V., SINUM 2010.

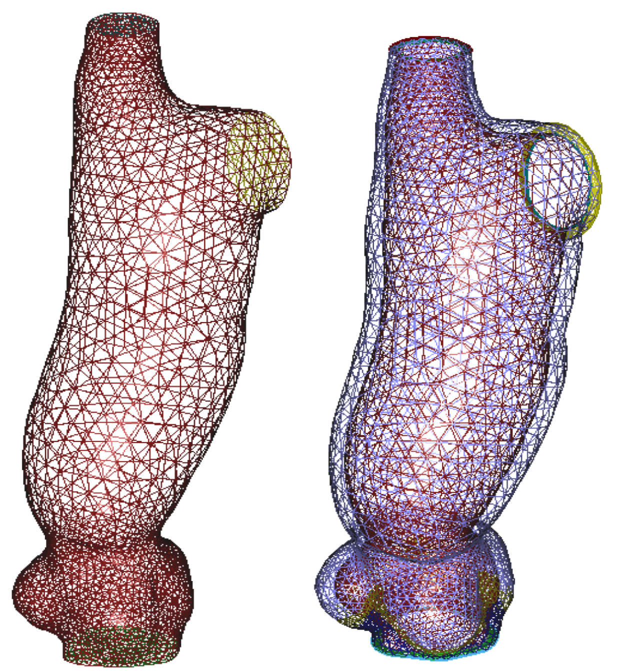
MOTIVATIONS

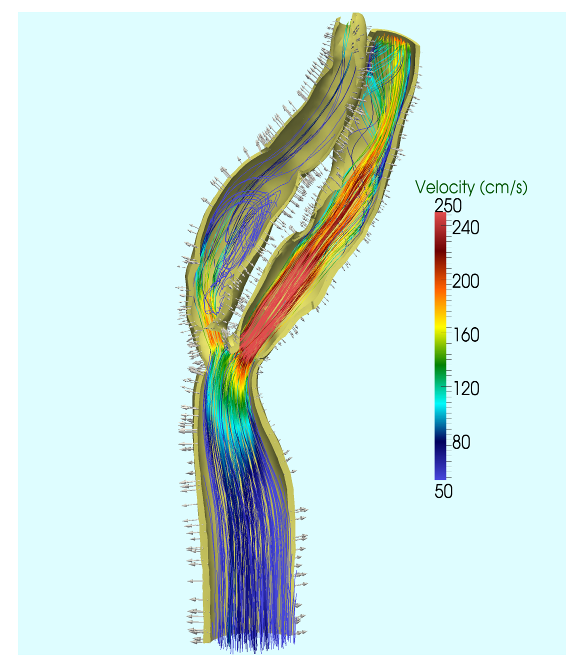


These works addressed flat interfaces

In some applications the interfaces are not flat,
for example they could be "cylindrical"

Example: Haemodynamics



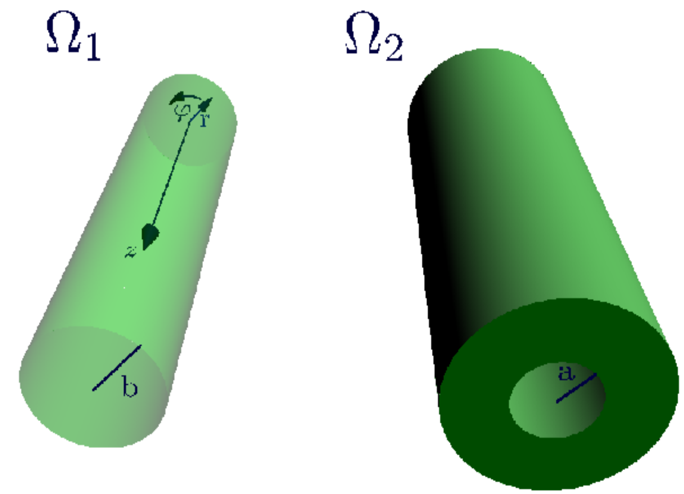


Aim of this work:
Extend the analysis and the optimization to cylindrical interfaces

THE DIFFUSION-REACTION PROBLEM (Gigante, Pozzoli, V., Submitted)

Consider the problem $-\triangle u + \eta u = f,\ x \in \Omega \equiv \mathbb{R}^3,\ \eta > 0,$ decompose Ω into two overlapping subdomains

$$\Omega_1 := \{(r, \varphi, z) : r < b,\ \varphi \in [0, 2\pi),\ z \in \mathbb{R}\},$$
$$\Omega_2 := \{(r, \varphi, z) : r > a,\ \varphi \in [0, 2\pi),\ z \in \mathbb{R}\}, \qquad 0 < a \leq b$$



Then, consider the Classical Schwarz Method at iteration n ($S_1 \to \infty,\ S_2 \to \infty$)

Given u_2^0 solve for $n \geq 0$ until convergence

1. The problem in the subdomain 1

$$\begin{cases} (\eta - \Delta_{cyl})\, u_1^n = f & \text{in } \Omega_1, \\ u_1^n = u_2^{n-1} & r = b,\ (\varphi, z) \in [0, 2\pi) \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \int_0^{2\pi} |u_1^n(r, \varphi, z)| d\varphi dz \text{ bounded} & \text{as } r \to 0^+, \\ u_1^n = 0 & \text{in } \{z = \pm\infty, r \leq b\}\,; \end{cases}$$

2. The problem in the subdomain 2

$$\begin{cases} (\eta - \Delta_{cyl})\, u_2^n = f & \text{in } \Omega_2, \\ u_2^n = u_1^n & r = a,\ (\varphi, z) \in [0, 2\pi) \times \mathbb{R}, \\ u_2^n = 0 & \text{in } \{r = +\infty\} \cup \{z = \pm\infty, r \geq a\}\,. \end{cases}$$

where $\Delta_{cyl} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$

CONVERGENCE OF THE CLASSICAL SCHWARZ METHOD

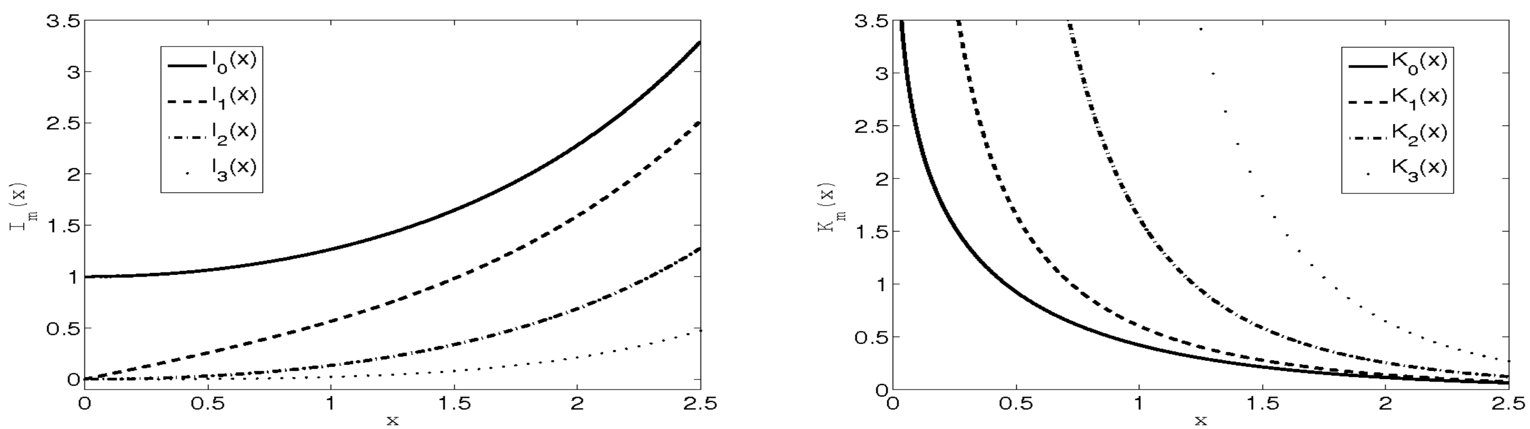
Introduce the Fourier transform in the cylindrical variables φ and z

$$\widehat{g}\left(r,m,k\right) = \mathcal{F}^{cyl}\left(g\right) := \int_{-\infty}^{+\infty}\int_{0}^{2\pi} g\left(r,\varphi,z\right)e^{-im\varphi}d\varphi e^{-ikz}dz,$$

where $m \in \mathbb{Z}$ and $k \in \mathbb{R}$ are the coordinates in the frequency domain. Applying it to the previous iterations, we obtain the following ODE's

$$\eta\,\widehat{u}_j^{\,n} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \widehat{u}_j^{\,n}}{\partial r}\right) + \frac{1}{r^2}m^2\widehat{u}_j^{\,n} + k^2\widehat{u}_j^{\,n} = 0 \qquad j = 1,2,$$

These are essentially modified Bessel equations whose solutions are $AI_m\left(\alpha r\right) + BK_m\left(\alpha r\right)$, for suitable coefficients A and B with $\alpha = \sqrt{k^2+\eta}$.
I_m and K_m are the Bessel functions of imaginary argument (Lebedev, 1972)



Proposition 1 *The reduction factor of the Classical Schwarz Method is*

$$\rho_{cla}^{cyl}(m,k) = \frac{I_m(\alpha a)}{I_m(\alpha b)}\frac{K_m(\alpha b)}{K_m(\alpha a)} \qquad \left(Flat\ \ case\text{: } \rho_{cla}^{flat}(k) = e^{-2\sqrt{k^2+\eta}(b-a)};\ Gander, 2006\right)$$

We have $\rho_{cla}^{cyl}(m,k) \leq 1$ with $\rho_{cla}^{cyl}(m,k) = 1$ iff $a = b$ (no overlap)

THE GENERALIZED SCHWARZ METHOD

Given u_2^0 solve for $n \geq 0$ until convergence

1. The problem in the subdomain 1

$$\begin{cases} (\eta - \Delta_{cyl})\, u_1^n = f & \text{in } \Omega_1, \\ \left(\mathcal{S}_1 + \frac{\partial}{\partial r}\right) u_1^n = \left(\mathcal{S}_1 + \frac{\partial}{\partial r}\right) u_2^{n-1} & r = b,\ (\varphi, z) \in [0, 2\pi) \times \mathbb{R}, \\ \int_{-\infty}^{\infty} \int_0^{2\pi} |u_1^n(r, \varphi, z)| d\varphi dz \text{ bounded} & \text{as } r \to 0^+, \\ u_1^n = 0 & \text{in } \{z = \pm\infty, r \leq b\}\,; \end{cases}$$

2. The problem in the subdomain 2

$$\begin{cases} (\eta - \Delta_{cyl})\, u_2^n = f & \text{in } \Omega_2, \\ \left(\mathcal{S}_2 + \frac{\partial}{\partial r}\right) u_2^n = \left(\mathcal{S}_2 + \frac{\partial}{\partial r}\right) u_1^n & r = a,\ (\varphi, z) \in [0, 2\pi) \times \mathbb{R}, \\ u_2^n = 0 & \text{in } \{r = +\infty\} \cup \{z = \pm\infty, r \geq a\}\,. \end{cases}$$

Proposition 2 *The reduction factor of the Generalized Schwarz Method is*

$$\rho^{cyl}(m, k) = \frac{\sigma_1 \left(-\frac{K_m(\alpha b)}{K_m'(\alpha b)}\right) - \alpha}{\sigma_1 \left(\frac{I_m(\alpha b)}{I_m'(\alpha b)}\right) + \alpha} \cdot \frac{\sigma_2 \left(\frac{I_m(\alpha a)}{I_m'(\alpha a)}\right) + \alpha}{\sigma_2 \left(-\frac{K_m(\alpha a)}{K_m'(\alpha a)}\right) - \alpha}$$

where $\sigma_j(m, k)$ *denote the symbols of* $\mathcal{S}_j$ *(Flat case:* $\rho^{flat}(k) = \frac{\sigma_1 - \alpha}{\sigma_1 + \alpha} \cdot \frac{\sigma_2 + \alpha}{\sigma_2 - \alpha} e^{-2\sqrt{k^2+\eta}(b-a)}$*; Gander, 2006)*

Proposition 3 *The Generalized Schwarz Method converges faster than the Classical Schwarz Method, provided that*

$$\sigma_1 > -\frac{1}{2}\alpha \left(\frac{I_m'(\alpha b)}{I_m(\alpha b)} + \frac{K_m'(\alpha b)}{K_m(\alpha b)}\right), \qquad \sigma_2 < -\frac{1}{2}\alpha \left(\frac{I_m'(\alpha a)}{I_m(\alpha a)} + \frac{K_m'(\alpha a)}{K_m(\alpha a)}\right). \tag{1}$$

In particular, under conditions (1), $\rho^{cyl}(m, k) < 1$ *for* $a = b$ *(no overlap)*

OPTIMIZATION FOR CYLINDRICAL INTERFACES

(Gigante, Pozzoli, V., Submitted)

Reduction factor:

$$\rho^{cyl}(m,k) = \frac{\sigma_2(m,k)\, I_m(\alpha a) + \alpha I'_m(\alpha a)}{\sigma_2(m,k)\, K_m(\alpha a) + \alpha K'_m(\alpha a)} \cdot \frac{\sigma_1(m,k)\, K_m(\alpha b) + \alpha K'_m(\alpha b)}{\sigma_1(m,k)\, I_m(\alpha b) + \alpha I'_m(\alpha b)}$$

$\longrightarrow$ The optimal choices of σ_1 and σ_2 are

$$\sigma^{cyl}_{1,opt}(m,k) = -\alpha \frac{K'_m(\alpha b)}{K_m(\alpha b)} = \sigma^{flat}_{1,opt}(k) \frac{K'_m(\alpha b)}{K_m(\alpha b)} > 0,$$

$$\sigma^{cyl}_{2,opt}(m,k) = -\alpha \frac{I'_m(\alpha a)}{I_m(\alpha a)} = \sigma^{flat}_{2,opt}(k) \frac{I'_m(\alpha a)}{I_m(\alpha a)} < 0.$$

providing a correction of the values obtained from the flat analysis (Gander, 2006):

$$\sigma^{flat}_{1,opt}(k) = \alpha = \sqrt{k^2+\eta}, \qquad \sigma^{flat}_{2,opt}(k) = -\alpha = -\sqrt{k^2+\eta}$$

These symbols give operators S_1 and S_2 that are difficult to use numerically $\longrightarrow$ 3 different approaches:

1) Constant approximations for localized frequencies

2) Second order approximation for localized frequencies

3) Uniformly optimized approximations

1) Constant approximations for localized frequencies

Evaluate $\sigma_{j,opt}^{cyl}$ for $k = k_0$, $m = m_0$:

$$\sigma_{1,T0}^{cyl}(m_0, k_0) = -\sqrt{k_0^2+\eta}\,\frac{K'_{m_0}\left(\sqrt{k_0^2+\eta}\,b\right)}{K_{m_0}\left(\sqrt{k_0^2+\eta}\,b\right)}, \quad \sigma_{2,T0}^{cyl}(m_0, k_0) = -\sqrt{k_0^2+\eta}\,\frac{I'_{m_0}\left(\sqrt{k_0^2+\eta}\,a\right)}{I_{m_0}\left(\sqrt{k_0^2+\eta}\,a\right)}.$$

Note: For $k_0 = m_0 = 0$ we obtain

$$\sigma_{1,T0}^{cyl}(0,0) = -\sqrt{\eta}\,\frac{K'_0\left(\sqrt{\eta}\,b\right)}{K_0\left(\sqrt{\eta}\,b\right)} > 0, \quad \sigma_{2,T0}^{cyl}(0,0) = -\sqrt{\eta}\,\frac{I'_0\left(\sqrt{\eta}\,a\right)}{I_0\left(\sqrt{\eta}\,a\right)} < 0.$$

Proposition 4 *If the only non-vanishing angular frequency is $m = 0 \rightarrow$*
i) For the Classical Schwarz Method with overlap $b - a = O(h)$ the maximum of the reduction factor has the following asymptotic behavior:

$$\max_{k_{min}\le k\le k_{max}} |\rho_{cla}^{cyl}(0,k)| = |\rho_{cla}^{cyl}(0,0)| = 1 - \sqrt{\eta}\left(\frac{K_1(\sqrt{\eta}a)}{K_0(\sqrt{\eta}a)} + \frac{I_1(\sqrt{\eta}a)}{I_0(\sqrt{\eta}a)}\right) h + O(h^2),$$

ii) For the Generalized Schwarz Method with constant approximations of the optimal symbols and without overlap, the maximum of the reduction factor behaves as

$$\max_{k_{min}\le k\le k_{max}} |\rho_{T0}^{cyl}(0,k,0,0)| = |\rho_{T0}^{cyl}(0,k_{max},0,0)| = 1 - \frac{2\sqrt{\eta}}{\pi}\left(\frac{K_1(\sqrt{\eta}a)}{K_0(\sqrt{\eta}a)} + \frac{I_1(\sqrt{\eta}a)}{I_0(\sqrt{\eta}a)}\right) h + O(h^2).$$

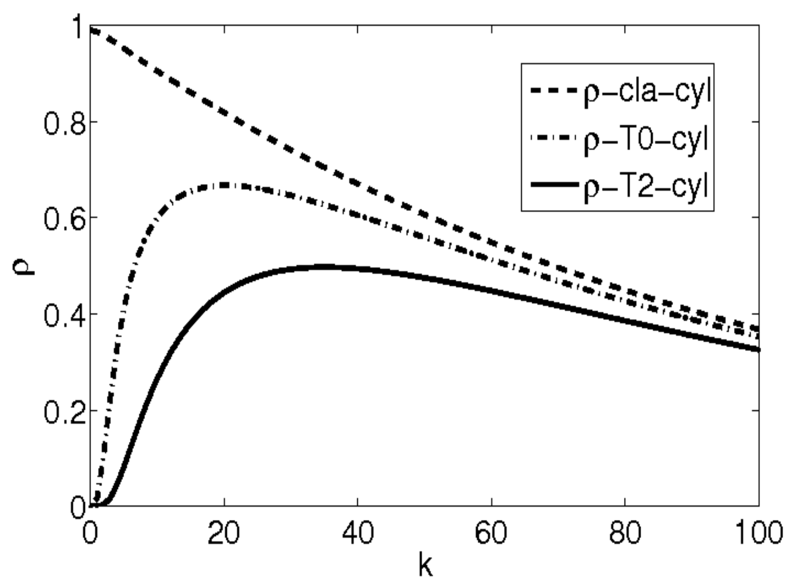
The asymptotic performance of the Classical Schwarz Method with overlap of the order of h is the same of the Generalized Schwarz Method with constant interface approximations of the optimal symbols and without overlap (as in the flat case, see Gander, 2006)

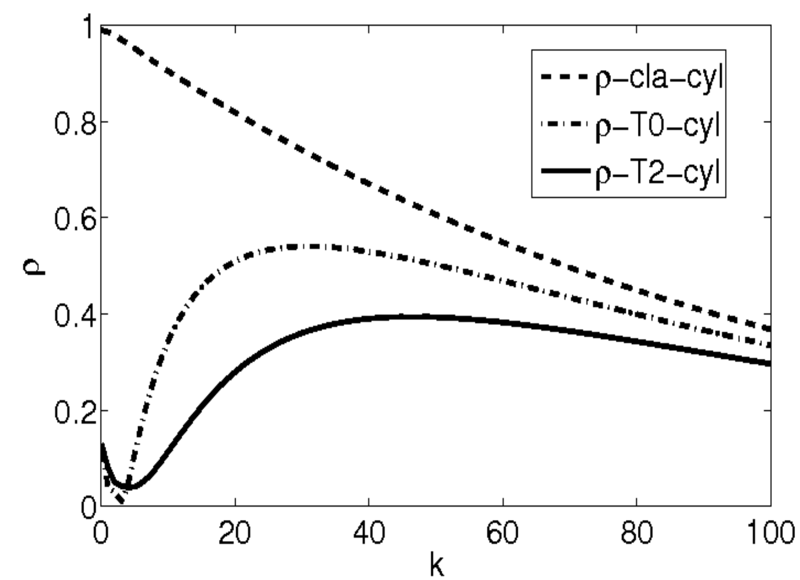
2) Second order approximation for localized frequencies k_0

$$\sigma^{cyl}_{1,T2}(m_0,k_0,k) = -\sqrt{k_0^2+\eta}\frac{K'_{m_0}\left(\sqrt{k_0^2+\eta}b\right)}{K_{m_0}\left(\sqrt{k_0^2+\eta}b\right)} + \frac{b}{2}\left[\left(\frac{K'_{m_0}\left(\sqrt{k_0^2+\eta}b\right)}{K_{m_0}\left(\sqrt{k_0^2+\eta}b\right)}\right)^2 - \left(1+\frac{m_0^2}{(k_0^2+\eta)b^2}\right)\right](k^2-k_0^2),$$

$$\sigma^{cyl}_{2,T2}(m_0,k_0,k) = -\sqrt{k_0^2+\eta}\frac{I'_{m_0}\left(\sqrt{k_0^2+\eta}a\right)}{I_{m_0}\left(\sqrt{k_0^2+\eta}a\right)} + \frac{a}{2}\left[\left(\frac{I'_{m_0}\left(\sqrt{k_0^2+\eta}a\right)}{I_{m_0}\left(\sqrt{k_0^2+\eta}a\right)}\right)^2 - \left(1+\frac{m_0^2}{(k_0^2+\eta)a^2}\right)\right](k^2-k_0^2).$$

Remark: These approximations hold also for $k_0 \neq 0$!





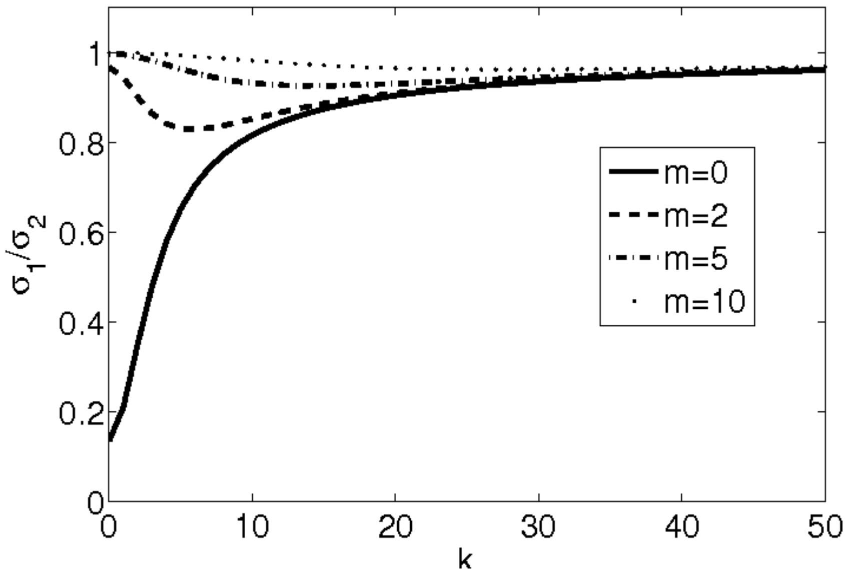
Reduction factors $m=0,\ k_0=0,\ \eta=1,\ a=0.495,\ b=0.5.$
Left: $m_0=0$; Right: $m_0=1$.

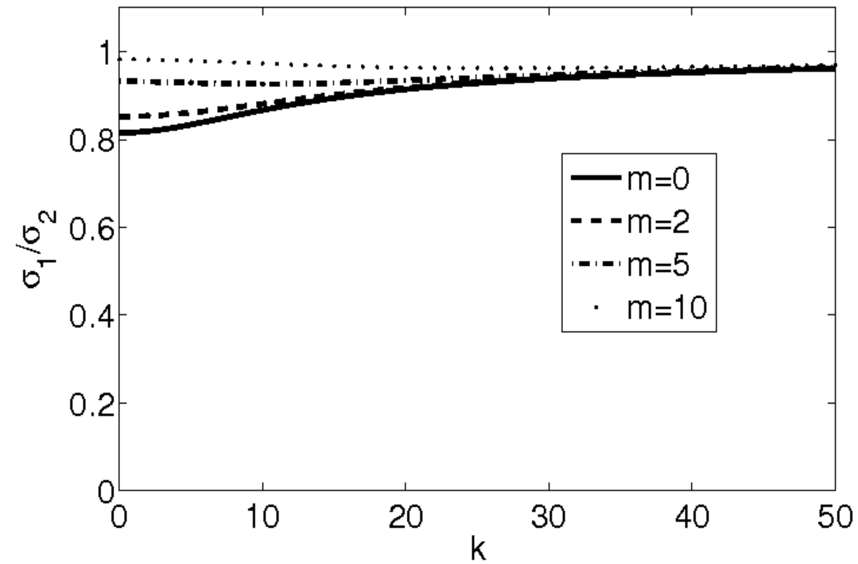
3) Uniformly optimized approximations

In the flat case, $|\sigma_{1,opt}^{flat}(k)| = |\sigma_{2,opt}^{flat}(k)|$ so that it made sense to look for optimized constant values of type (Gander, SINUM 2006)

$$\sigma_{1,OO0}^{flat} = -\sigma_{2,OO0}^{flat} = p$$

In the cylindrical case $|\sigma_{1,opt}^{cyl}(k)| \neq |\sigma_{2,opt}^{cyl}(k)|$!





Ratio between the optimal interface symbols as a function of k. $a = b = 0.5$

$\eta = 1$ $\eta = 100$

The ratio between the optimal symbols is almost equal to 1 (apart when η, m and k are all small) $\to$ It makes sense to look for optimized constant values of type

$$\sigma_{1,OO0}^{cyl} = -\sigma_{2,OO0}^{cyl} = p$$

3) Uniformly optimized approximations (cont'd)

Set

$$A\left(m,k\right)=\sqrt{\eta+k^2}\frac{I'_m\left(a\sqrt{\eta+k^2}\right)}{I_m\left(a\sqrt{\eta+k^2}\right)},\qquad B\left(m,k\right)=-\sqrt{\eta+k^2}\frac{K'_m\left(a\sqrt{\eta+k^2}\right)}{K_m\left(a\sqrt{\eta+k^2}\right)}.$$

$$A_-:=A\left(m_{min},k_{min}\right),\qquad B_-:=B\left(m_{min},k_{min}\right),$$

$$A_+:=A\left(m_{max},k_{max}\right),\qquad B_+:=B\left(m_{max},k_{max}\right),$$

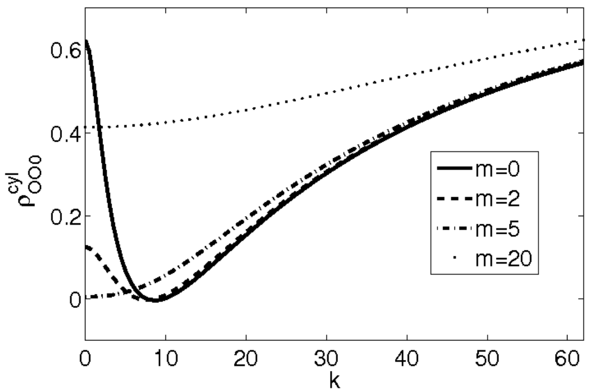
Theorem 1 *Assume $B_-\leq A_+$ and no overlap. If*

$$p_{opt}=\sqrt{\frac{A_+B_+\left(A_-+B_-\right)-A_-B_-\left(A_++B_+\right)}{A_++B_+-A_--B_-}},$$

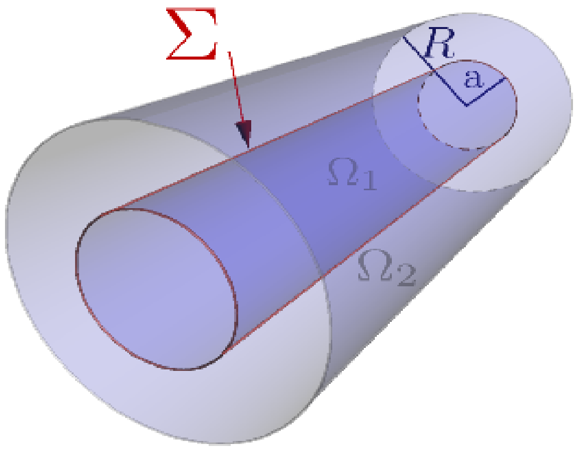
then

$$\bar{\rho}^{cyl}_{OO0}:=\min_{p\geq 0}\max_{\substack{m\in[m_{min},m_{max}]\\ k\in[k_{min},k_+]}}\rho^{cyl}_{OO0}\left(m,k,p\right)=\rho^{cyl}_{OO0}\left(m_{min},k_{min},p_{opt}\right)=\rho^{cyl}_{OO0}\left(m_{max},k_{max},p_{opt}\right).$$

Note: The optimization has been performed for the function $\rho^{cyl}_{OO0}\left(m,k,p\right)$ which is not necessarily positive. However, for the values considered in this work, the negative part featured always small absolute values



NUMERICAL RESULTS (Gigante, Pozzoli, V., Submitted)



Numerical experiments: No overlap, $a = 0.5$, $L = 5$, $R = 1$, $f = u = 0$, Finite Element Library **LIFEV** (www.lifev.org),

Test 1: Cylindrical asymmetry (only $m = 0$ involved). Initial condition:

$$\begin{cases} u_2^0 = \dfrac{1}{e^4}\left(\dfrac{z^2 - 2.5}{6.25}\right)^3 & \text{on } \Sigma, \\ \dfrac{\partial u_2^0}{\partial n} = 0 & \text{on } \Sigma. \end{cases}$$

σ/η	0.1	1	10
$\sigma_{1,T0}^{flat}(0,0)$	0.32	1.00	3.16
$\sigma_{2,T0}^{flat}(0,0)$	-0.32	-1.00	-3.16
$\sigma_{1,T0}^{cyl}(0,0)$	0.98	1.79	4.06
$\sigma_{2,T0}^{cyl}(0,0)$	-0.02	-0.24	-1.95

Values of the constant interface approximations

σ/η	0.1	1	10
$\sigma_{T0}^{flat}(0,0)$	160	92	55
$\sigma_{T0}^{cyl}(0,0)$	28	35	39

Number of iterations for different values of η by using the constant interface approximations

NUMERICAL RESULTS (cont'd)

	σ_1	σ_2	num iter
$\sigma_{T0}^{cyl}(0,0)$	1.79	-0.24	35
$\sigma_{T0}^{cyl}(0,1)$	2.24	-0.47	35
$\sigma_{T0}^{cyl}(0,5)$	6.03	-3.93	28
$\sigma_{T0}^{cyl}(0,10)$	11.01	-8.98	21
$\sigma_{T0}^{cyl}(0,15)$	16.00	-13.99	30
$\sigma_{T0}^{cyl}(0,30)$	31.00	-29.00	58
$\sigma_{T0}^{cyl}(0,60)$	61.00	-59.00	111

Values of the constant interface parameters (left) and number of iterations (right) for different values of k_0. $m_0 = 0$.

p	# iter
8.79	25

Number of iterations by using the optimized constant interface parameter

There exists an optimal value of k_0 which guarantees the best convergence. However, it is difficult to estimate it a priori $\longrightarrow$ Optimized constant parameter

NUMERICAL RESULTS (cont'd)

Test 2: Non-null localized angular frequency ($m = 2$).
Exact solution $u = (x^2 - y^2)z,\ f = \eta\, u$, initial condition

$$\begin{cases} u_2^0 = 5(x^2 - y^2)\sin(\pi z/5) & \text{on } \Sigma, \\ \dfrac{\partial u_2^0}{\partial n} = 0 & \text{on } \Sigma. \end{cases}$$

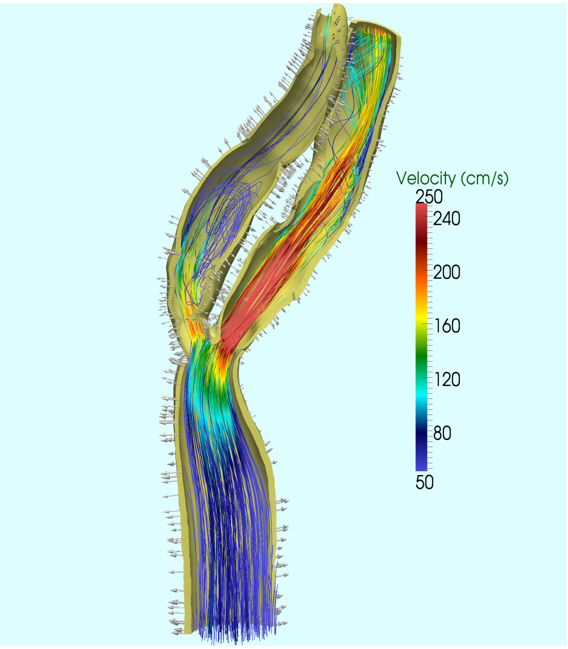
	σ_1	σ_2	num iter
$\sigma_{T0}^{cyl}(0,0)$	1.79	-0.24	283
$\sigma_{T0}^{cyl}(2,0)$	4.22	-4.08	60

Values of the constant interface parameters (left) and number of iterations (right)

Big improvement if the constant parameters are evaluated for $m_0 = 2$!

THE FLUID-STRUCTURE INTERACTION PROBLEM

(Gigante, V., In preparation)



We consider a fluid flowing in an elastic channel.

Simplified models:

- Incompressible, linear and non-viscous fluid
- Wave equation

Fluid domain: $\Omega_f := \{r < R, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$

Structure domain: $\Omega_s := \{r > R, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$

$$\begin{cases} \rho_f \delta_t \boldsymbol{u} + \nabla_{cyl} p = \boldsymbol{0} & \text{in } \Omega_f, \\ \nabla_{cyl} \cdot \boldsymbol{u} = 0 & \text{in } \Omega_f, \\ \int_{-\infty}^{\infty} \int_0^{2\pi} |\zeta^{j+1}(r, \varphi, z)| d\varphi dz \text{ bounded} & \text{as } r \to 0^+, \ \zeta = p, \boldsymbol{u}, \\ u_r = \delta_t \eta_r & \text{on } \Sigma, \\ -p\boldsymbol{n} = \lambda \nabla_{cyl} \boldsymbol{\eta}\, \boldsymbol{n} & \text{on } \Sigma, \\ \eta_\theta = \eta_z = 0 & \text{on } \Sigma \\ \rho_s \delta_{tt} \boldsymbol{\eta} - \lambda \triangle_{cyl} \boldsymbol{\eta} = \boldsymbol{0} & \text{in } \Omega_s, \\ \boldsymbol{\eta} = \boldsymbol{0} & \text{in } \{r \to \infty\} \cup \{z = \pm\infty, r > R\}, \end{cases}$$

where $\delta_t w := \frac{w - w^n}{\Delta t}$, $\delta_{tt} w := \frac{\delta_t w - \delta_t w^n}{\Delta t}$

Coupling conditions: Continuity of velocities and stresses

GENERALIZED SCHWARZ METHOD - FSI

Given $\boldsymbol{u}^0$, η^0 and two linear operators $S_f \neq S_s$, solve for $j \geq 0$ until convergence

1. Fluid problem

$$\begin{cases} \rho_f \delta_t \boldsymbol{u}^{j+1} + \nabla_{cyl} p^{j+1} = \mathbf{0} & \text{in } \Omega_f, \\ \nabla_{cyl} \cdot \boldsymbol{u}^{j+1} = 0 & \text{in } \Omega_f, \\ \int_{-\infty}^{\infty} \int_0^{2\pi} |\zeta^{j+1}(r,\varphi,z)| d\varphi dz \text{ bounded} & \text{as } r \to 0^+,\ \zeta = p, \boldsymbol{u}, \\ S_f \Delta t\, \delta_t u_r^{j+1} - p^{j+1} = \dfrac{S_f}{\Delta t} \eta_r^j + \lambda\, \partial_r \eta_r^j + F_1(u_r^n, \eta_r^n, \eta_r^{n-1}) & \text{on } \Sigma; \end{cases}$$

2. Structure problem

$$\begin{cases} \rho_s \delta_{tt} \boldsymbol{\eta}^{j+1} - \lambda \triangle_{cyl} \boldsymbol{\eta}^{j+1} = \mathbf{0} & \text{in } \Omega_s, \\ \boldsymbol{\eta}^{j+1} = \mathbf{0} & \text{in } \{r \to \infty\} \cup \{z = \pm\infty, r > R\}, \\ \dfrac{S_s}{\Delta t} \eta_r^{j+1} + \lambda\, \partial_r \eta_r^{j+1} = S_s \Delta t\, \delta_t u_r^{j+1} - p^{j+1} + F_2(u_r^n, \eta_r^n, \eta_r^{n-1}) & \text{on } \Sigma, \\ \eta_\theta^{j+1} = \eta_z^{j+1} = 0 & \text{on } \Sigma \end{cases}$$

Proposition 5 *The reduction factor of the previous iterations is given by*

$$\rho^j(m,k) = \left| \frac{\sigma_f\, K_m(\beta R) + \lambda \Delta t \beta\, K_m'(\beta R)}{\sigma_s\, K_m(\beta R) + \lambda \Delta t \beta\, K_m'(\beta R)} \cdot \frac{\rho_f\, I_m(kR) + \sigma_s \Delta t\, k\, I_m'(kR)}{\rho_f\, I_m(kR) + \sigma_f \Delta t\, k\, I_m'(kR)} \right|, \quad \beta := \sqrt{k^2 + \frac{\rho_s}{\lambda \Delta t^2}}.$$

For the Dirichlet-Neumann scheme, that is for $\sigma_f \to \infty$ and $\sigma_s = 0$, we obtain

$$\rho_{DN}(m,k) = \left| \frac{\rho_f}{\beta} \frac{K_m(\beta R)}{\lambda \Delta t\, K_m'(\beta R)} \cdot \frac{I_m(kR)}{\Delta t\, k\, I_m'(kR)} \right|.$$

$\rightarrow$ slow or even no convergence for $\rho_f \simeq \rho_s$ (high added mass effect)

OPTIMIZATION - FSI

(Gigante, V., In preparation)

The optimal values are given by

$$\sigma_f^{opt}(m,k) = -\frac{\lambda \Delta t \beta K_m'(\beta R)}{K_m(\beta R)} > 0, \quad \sigma_s^{opt}(m,k) = -\frac{\rho_f I_m(kR)}{\Delta t k I_m'(kR)} < 0$$

In this case $|\sigma_f| \neq |\sigma_s|$ in general for any η, m and k
$\longrightarrow$ we can not anymore look for the same constant optimed parameter!

Idea: Look for two function $\widehat{\sigma}_f(p)$ and $\widehat{\sigma}_s(p)$ approximating the optimal values and depending only on one parameter p

Exploiting the properties of the Bessel functions, we have

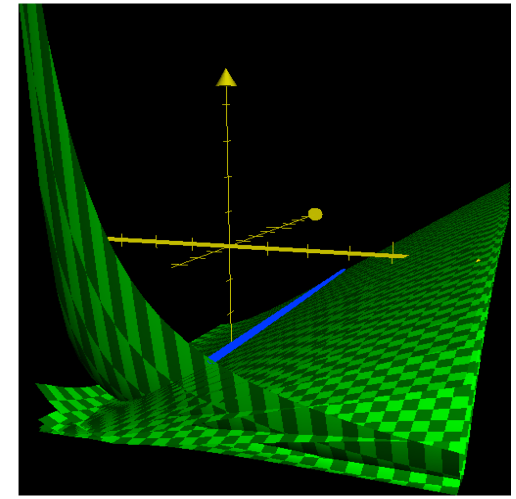
$$\left(\frac{R}{\lambda \Delta t}\sigma_f^{opt}(m,k) - \frac{1}{2}\right)^2 - R^2 \frac{\rho_s}{\lambda \Delta t^2} \simeq \left(-\frac{R}{\Delta t}\frac{1}{\sigma_s^{opt}(m,k)} + \frac{1}{2}\right)^2$$

By forcing that the latter is satisfied exactly, we obtain an approximated relationship between $\widehat{\sigma}_f$ and $\widehat{\sigma}_s$
Setting $\widehat{\sigma}_s(p) = -p$, we get

$$\widehat{\sigma}_f(p) = \lambda\left(\sqrt{\left(\frac{1}{p} + \frac{\Delta t}{2R}\right)^2 + \frac{\rho_s}{\lambda}} + \frac{\Delta t}{2R}\right).$$

OPTIMIZATION - FSI (cont'd)

The analysis to determine the optimal value of p is on going...
In the meantime we evaluated a possible optimized
value of p by drawing a plot of ρ



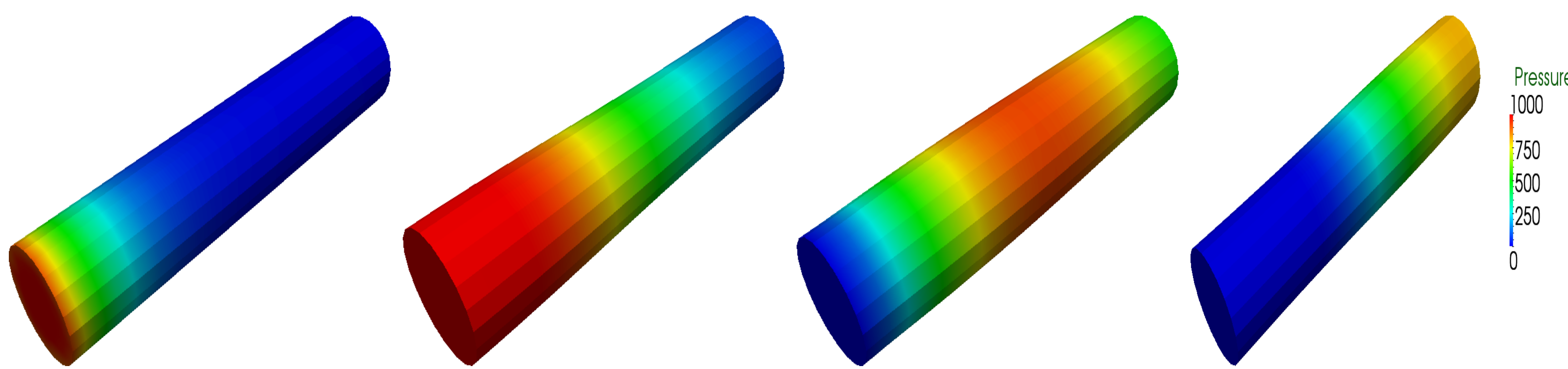
ρ vs k and p

Test FSI:
Fluid: Incompressible Navier-Stokes equations, Structure: Linear elasticity

$a = 0.5,\ L = 5,\ \Delta t = 0.001,\ P_{in} = \begin{cases} 1000 & t < 0.008 \\ 0 & t \geq 0.008 \end{cases}$ $\rightarrow$ estimated $p = 969 \rightarrow \widehat{\sigma}_f = 2762,\ \widehat{\sigma}_s = -969$

H_s	0.1	0.2	0.5	1.0
	13.5	8.1	5.6	6.8

Number of iterations for different values of the structure thickness H_s by using the estimated optimized constant parameter



$t = 0.001$ $t = 0.004$ $t = 0.009$ $t = 0.013$