Discontinuous Galerkin Isogeometric Analysis for Elliptic PDEs on Surfaces

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NFN Project S117-03

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DD22, Lugano, Switzerland, September 16 - 20, 2013.

Outline

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Motivation - PDEs on Manifolds

Discontinuous Galerkin Isogeometric Analysis (DGIGA)

- IGA provides a bridge between numerical simulation of physical phenomena and Computer Aided Design (CAD).
- IGA uses the same basis functions for approximation and geometry description as in CAD (based on NURBS).
- IGA yield **patches** which play the role of **subdomains**.
- The need to assemble such subdomains. E.g: automobiles have about 3000 parts.

PDEs on surfaces arise in various areas, for instance:

- Mechanics: shell problems, etc..
- Cell biology: phase separation on biomembranes, diffusion processes on plasma membranes,etc.
- Image processing on surfaces : optical flow on evolving surfaces.

Motivation - Numerical Approaches

Numerical methods are based on:

- **PDE+FEM**: $\mathbf{9.48} \times \mathbf{10^6}-$Google Hits in $\sim 0.24s$ (1950's).
  - SurfacePDE+FEM: Steger (1983), Baumgardner+Frederickson (1985), Dziuk (1988),..., Survey by Dziuk and Elliot (2013).
  - SurfacePDE+FEM+MGM: Kornhuber+Yserentant (2008), Landsberg+Voigt (2010),...
- **PDE+FEM+DG**: $\mathbf{0.805} \times \mathbf{10^6}-$ Hits in $\sim 0.26s$ (1970's)
  - SurfacePDE+DG+FEM: Dedner+Madhavan+Stinner (2013).
- **PDE+IGA**: $\mathbf{0.0475} \times \mathbf{10^6}-$ Hits in $\sim 0.20s$.
  - Hughes et al.(2005, 2006, 2009)...
  - SurfacePDE+IGA: Dede+Quarteroni (Preprint, 2012).
  - DG+IGA: Brunero (2012), Master thesis supervised by Langer, Pechstein (Linz) and Pavarino (Milan)
  - SurfacePDE+DG+IGA: **in this talk** !

Preliminaries

- Manifold: $\Omega \subset \mathbb{R}^3$ is compact, connected, oriented and sufficiently smooth.
- $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega^{(i)}}, \quad \Omega^{(i)} \cap \Omega^{(j)} = \emptyset$ for $i \neq j$.
- $\mathcal{T}_H = \{\Omega^{(i)}\}_{i=1}^{N}$,
- Parameter domain: $\widehat{\Omega} \subset \mathbb{R}^2$.
- Geometrical mapping:

$$G^{(i)} : \widehat{\Omega} \to \Omega^{(i)} \subset \mathbb{R}^3, \quad \xi = (\xi_1, \xi_2) \mapsto G^{(i)}(\xi).$$

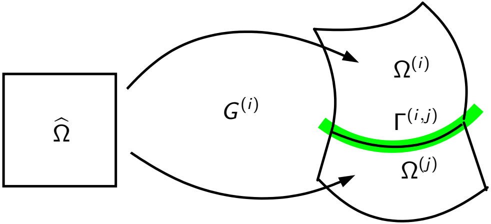


Figure : Isogeometric mapping .

Geometrical mapping by NURBS

- Knot Vector: $\Xi = (\xi_1, \ldots, \xi_{n+p+1})$ with $\xi_k \leq \xi_{k+1}$.
  - p - polynomial order.
  - n number of basis functions.
  - Open if first and last knot value appear $p+1$ times.
- $\widehat{N}^{(i)}_{k,p} : (0,1) \to \mathbb{R}, k = 1, \ldots, n.$
- Univariate B-Spline basis function: Cox-de Boor recurssion

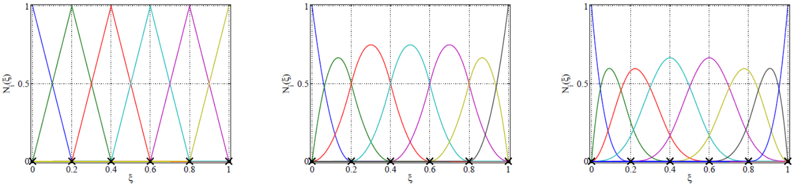
$$\widehat{N}^{(i)}_{k,0}(\xi) = \begin{cases} 1 & \text{if} \quad \xi_k \leq \xi < \xi_{k+1} \\ 0 & \text{else} \end{cases}, \text{ and for } p \geq 1$$

$$\widehat{N}^{(i)}_{k,p}(\xi) = \frac{\xi - \xi_k}{\xi_{k+p} - \xi_k} \widehat{N}^{(i)}_{k,p-1}(\xi) + \frac{\xi_{k+p+1} - \xi}{\xi_{k+p+1} - \xi_{k+1}} \widehat{N}^{(i)}_{k+1,p-1}(\xi).$$

- basis functions of order p have $p - m$ continuous derivatives across knots, where m is multiplicity of knot.

Example of Knot Vector

$$\Xi = \left\{ \underbrace{0, \ldots, 0}_{p+1}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \underbrace{1, \ldots, 1}_{p+1} \right\}$$



$p = 1$ $\quad$ $p = 2$ $\quad$ $p = 3$

Figure : B-spline basis functions of order $p = 1, 2$ and 3.

Geometrical mapping by NURBS

- bivariate NURBS basis functions:

$$\widehat{R}^{(i)}_{(\mathbf{k},\mathbf{p})}(\xi_1,\xi_2) := \frac{\widehat{R}^{N}_{(\mathbf{k},\mathbf{p})}(\xi_1,\xi_2)}{\widehat{R}^{D}(\xi_1,\xi_2)}$$

- where

$$\begin{aligned}\widehat{R}^{N}_{(\mathbf{k},\mathbf{p})}(\xi_1,\xi_2) &= \widehat{N}^{(i)}_{k_1,p_1}(\xi_1)\widehat{N}^{(i)}_{k_2,p_2}(\xi_2)w_{(k_1,k_2)}\\ \widehat{R}^{D}(\xi_1,\xi_2) &= \sum_{\mathbf{k}} \widehat{R}^{N}_{(\mathbf{k},\mathbf{p})}(\xi_1,\xi_2),\end{aligned}$$

$\mathbf{k} = (k_1,k_2), \mathbf{p} = (p_1,p_2)$ and the weights $w_{\mathbf{k}} > 0$.

Outline

The Diffusion Problem

Find the weak solution u of the boundary value problem

$$\begin{cases} -\nabla_\Omega \cdot (\alpha \nabla_\Omega u) = f \quad in \;\; \Omega \\ \qquad\qquad\qquad u = g_D \quad on \, \Gamma_D \\ \qquad \alpha \nabla_\Omega u \cdot \mathbf{n} = g_N \quad on \, \Gamma_N \end{cases} \tag{1}$$

where

- $\Omega \subset \mathbb{R}^3 -$ compact, connected, oriented and sufficiently smooth surface (2D manifold).
- $\alpha \in L^\infty(\Omega)$, uniformly positive.
- $f \in L^2(\Omega), g_D \in H^{\frac{1}{2}}(\Gamma_D), g_N \in L^2(\Gamma_N)$.
- $\mathbf{n} -$ unit outward normal vector in $\partial\Omega$.
- $\partial\Omega = \Gamma_D \cup \Gamma_N$.
- $\nabla_\Omega -$ surface gradient.

Examples of Surfaces

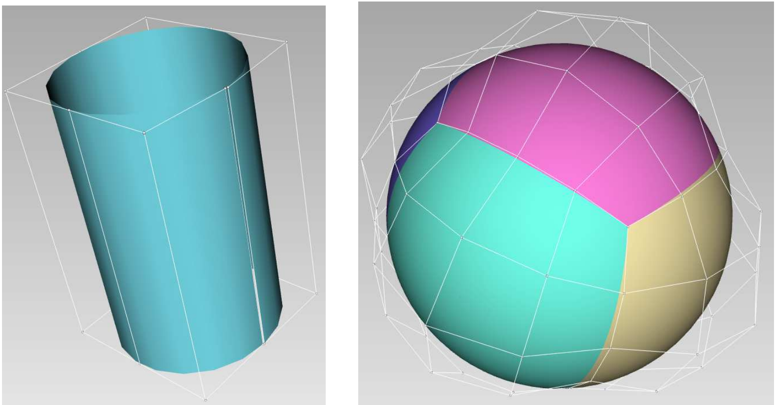


Figure : NURBS of an open surface ($\mathrm{meas}(\partial\Omega) \neq 0$) cylinder (left) and a closed surface($\mathrm{meas}(\partial\Omega) = 0$) sphere (right). The "mesh elements" and control points are highlighted.

Elliptic PDE on the Surface

- Considering

$$G^{(i)} : \widehat{\Omega} \subset \mathbb{R}^2 \to \Omega^{(i)} \subset \mathbb{R}^3, \quad i = 1, \dots, N.$$

$$\psi(\mathbf{x}) = \widehat{\psi}(\xi) \circ G^{(i)-1}(\xi), \quad \widehat{\psi}(\xi) = \psi(G^{(i)}(\xi)), \quad \psi \in C^0(\Omega).$$

- Projection operator:

$$\mathbf{P}(\mathbf{x}) = I - \mathbf{n}_\Omega(\mathbf{x}) \otimes \mathbf{n}_\Omega(\mathbf{x}) \quad \textit{for} \quad \mathbf{x} \in \Omega,$$

- Gradient on manifold:

$$\begin{aligned} \nabla_\Omega \psi(\mathbf{x}) &:= \nabla \widetilde{\psi}(\mathbf{x}) - \left(\nabla \widetilde{\psi}(\mathbf{x}) \cdot \mathbf{n}_\Omega(\mathbf{x}) \right) \mathbf{n}_\Omega(\mathbf{x}) \\ &:= \left[\mathbf{P}(\mathbf{x}) \nabla \widetilde{\psi}(\mathbf{x}) \right] \quad \textit{for} \quad \widetilde{\psi} \subset \mathbb{R}^3 \end{aligned}$$

- Divergence on manifold:

$$\nabla_\Omega \cdot \psi(\mathbf{x}) := \mathrm{trace}\left[\nabla_\Omega \psi(\mathbf{x}) \right]$$

- Laplace-Beltrami operator:

$$\Delta_\Omega \psi(\mathbf{x}) := \nabla_\Omega \cdot (\nabla_\Omega \psi(\mathbf{x})) := \mathrm{trace}\left[\mathbf{P}(\mathbf{x}) \nabla^2 \widetilde{\psi}(\mathbf{x}) \mathbf{P}(\mathbf{x}) \right]$$

Outline

Discontinuous Galerkin Formulation

- $\alpha_{|\Omega^{(i)}} = \alpha_i = \text{const}, \forall i = 1, \ldots, N$ (for simplicity).
- Sobolev spaces on manifolds: For any non-negative real number $s = \lfloor s \rfloor + \sigma$, for $\sigma \in [0, 1)$.

$$H^s(\Omega) := \left\{ v \in H^k(\Omega) : |\partial_\Omega^\kappa v|_{H^\sigma(\Omega)} < \infty \text{ for } |\kappa| = \lfloor s \rfloor \right\},$$

with corresponding norm

$$\|v\|_{H^s(\Omega)} := \left(\|v\|^2_{H^k(\Omega)} + \sum_{|\kappa| = \lfloor s \rfloor} |\partial_\Omega^\kappa v|^2_{H^\sigma(\Omega)} \right)^{1/2}.$$

- broken Sobolev space

$$\mathcal{V} := H^s(\mathcal{T}_H) = \{ v \in L^2(\Omega) : v_{|\Omega^{(i)}} \in H^s(\Omega^{(i)}), \forall i = 1, \ldots, N \},$$

with corresponding broken Sobolev norm

$$\|v\|_{H^s(\mathcal{T}_H)} = \left(\sum_{\Omega^{(i)} \in \mathcal{T}_H} \|v\|^2_{H^s(\Omega^{(i)})} \right)^{\frac{1}{2}}.$$

Derivation of the DG Variational Formulation

- Multiply (1) by a test function $v \in H^{1+s}(\mathcal{T}_H)$ with $s > 1/2$, integrate on each patch.
- Apply **Green's theorem** on each patch.
- Sum over all patches.
- Consider all boundary types

$$\Gamma^I = \partial\Omega^{(i)} \cap \partial\Omega^{(j)}, \quad \mathcal{E}_I := \{\Gamma^I : i > j, meas_1(\Gamma^I) > 0\}.$$
$$\Gamma^D = \partial\Omega^{(i)} \cap \Gamma_D, \quad \mathcal{E}_D := \{\Gamma^D : i = 1, \ldots, N\}.$$
$$\Gamma^N = \partial\Omega^{(i)} \cap \Gamma_N, \quad \mathcal{E}_N := \{\Gamma^N : i = 1, \ldots, N\}.$$

  and $\mathcal{E}_I \cup \mathcal{E}_D \cup \mathcal{E}_N = \bigcup_{i=1}^N \partial\Omega^{(i)}$.
- Averages $\{\cdot\}$ and Jumps $[\cdot]$
- $\mathbf{n}$ is defined on $\bigcup_{i=1}^N \partial\Omega^{(i)}$ such that $\mathbf{n} = \mathbf{n}_i$ on Γ^I for $i > j$.

- Rewrite the sum over the interface integrals
- For smooth exact solution $[u] := 0 =: [u - g_D]$, add term for adjoint consistency with β.

$$\beta = -\mathbf{1}\,(\textbf{Symmetric}) \quad \text{and} \quad \beta = 1(Non - symmetric).$$

- add penalty term to penalize discontinuities $u \in H^{1+s}(\mathcal{T}_H)$ with penalty parameter $\delta^{(i,j)} = \delta/h$ and $\delta > 0$.

DG Variational Formulation

- Finally, we obtain the DG bilinear form

$$a_{DG}(u,v) := \sum_{i=1}^{N} \int_{\Omega^{(i)}} \alpha_i \nabla_\Omega u \cdot \nabla_\Omega v - \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \int_\Gamma \{\alpha \nabla_\Omega u \cdot \mathbf{n}\}[v]$$

$$+ \beta \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \int_\Gamma \{\alpha \nabla_\Omega v \cdot \mathbf{n}\}[u]$$

$$+ \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \alpha_{(i,j)} \delta^{(i,j)} \int_\Gamma [u][v]$$

- and the DG linear functional

$$\langle F, v\rangle := \int_\Omega fv + \sum_{\Gamma^N \in \mathcal{E}_N} \int_{\Gamma^N} g_N v$$

$$+ \sum_{\Gamma^D \in \mathcal{E}_D} \int_{\Gamma^D} \left(\beta \alpha_i \nabla_\Omega v \cdot \mathbf{n} + \alpha_i \delta^{(i,D)} v \right) g_D .$$

Discontinuous Galerkin Discretization

- Single patch NURBS:

$$V_h = \mathrm{span}\{R_{\mathbf{k}}\}_{\mathbf{k}}, \quad R_{\mathbf{k}} = \widehat{R}_{\mathbf{k}} \circ G^{-1}$$

  - $u_h \in V_h$ is represented as

$$u_h(\xi) = \sum_{\mathbf{k}} u_{\mathbf{k}} R_{\mathbf{k}}(\xi).$$

- Multi-patch NURBS:

$$\mathcal{V}_h = \{v \in L^2(\Omega) : \forall i = 1, \ldots, N \quad v_{|\Omega^{(i)}} \in V_h^i(\Omega^{(i)})\},$$

where

$$V_h^i = \mathrm{span}\{R_{\mathbf{k}}^{(i)}\}_{\mathbf{k}} \quad \text{and} \quad R_{\mathbf{k}}^{(i)} = \widehat{R}_{\mathbf{k}}^{(i)} \circ G^{(i)-1}.$$

Discontinuous Galerkin Discretization: meas$(\partial\Omega) \neq 0$.

Find $u_h \in \mathcal{V}_h$ such that

$$a_{DG}(u_h, v_h) = \langle F_{DG}, v_h \rangle, \quad \forall v_h \in \mathcal{V}_h, \tag{2}$$

where

$$\begin{aligned} a_{DG}(u_h, v_h) := & \sum_{i=1}^{N} \int_{\Omega^{(i)}} \alpha_i \nabla_\Omega u_h \cdot \nabla_\Omega v_h - \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \int_\Gamma \{\alpha \nabla_\Omega u_h \cdot \mathbf{n}\}[v_h] \\ & - \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \int_\Gamma \{\alpha \nabla_\Omega v_h \cdot \mathbf{n}\}[u_h] + \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \alpha_{(i,j)} \delta^{(i,j)} \int_\Gamma [u_h][v_h] \end{aligned}$$

$$\begin{aligned} \langle F_{DG}, v_h \rangle := & \int_\Omega f_h v_h + \sum_{\Gamma^N \in \mathcal{E}_N} \int_{\Gamma^N} g_N v_h \\ & + \sum_{\Gamma^D \in \mathcal{E}_D} \int_{\Gamma^D} \left(-\alpha_i \nabla_\Omega v_h \cdot \mathbf{n} + \alpha_i \delta^{(i,D)} v_h \right) g_D \,. \end{aligned}$$

Discontinuous Galerkin Discretization: $\text{meas}(\partial\Omega) = 0$.

Find $u_h \in \mathcal{V}_h$ satisfying $\int_\Omega u_h = 0$ such that

$$a_{DG}(u_h, v_h) = \langle F_{DG}, v_h \rangle, \quad \forall v_h \in \mathcal{V}_h, \tag{3}$$

where

- $a_{DG}(u_h, v_h)$ is the discretized symmetric bilinear form

$$\begin{aligned} a_{DG}(u_h, v_h) := \sum_{i=1}^{n} \int_{\Omega^{(i)}} \alpha_i \nabla_\Omega u_h \cdot \nabla_\Omega v_h - \sum_{\Gamma \in \mathcal{E}_I} \int_\Gamma \{\alpha \nabla_\Omega u_h \cdot \mathbf{n}\}[v_h] \\ - \sum_{\Gamma \in \mathcal{E}_I} \int_\Gamma \{\alpha \nabla_\Omega v_h \cdot \mathbf{n}\}[u_h] + \sum_{\Gamma \in \mathcal{E}_I} \alpha_{(i,j)} \delta^{(i,j)} \int_\Gamma [u_h][v_h] \end{aligned}$$

- $\langle F_{DG}, v_h \rangle := \int_\Omega f v_h$ is the linear functional satisfying $\int_\Omega f = 0$.

Necessary Ingredients for Analysis

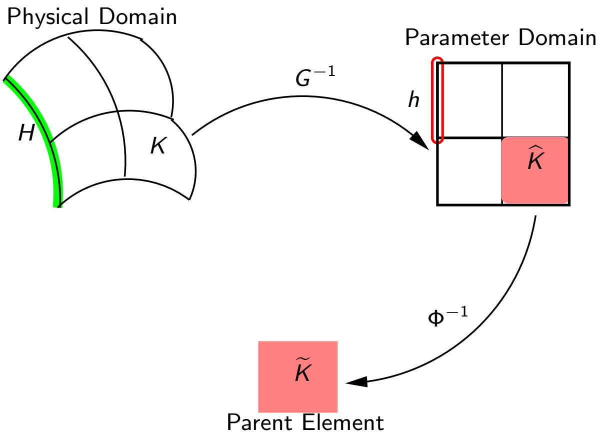


Figure : Schematic illustration of NURBS paraphernalia for a one-patch surface model.

Necessary Ingredients for Analysis

Lemma (IGA trace inequality, L+M)

There exists a generic positive constant $C_{shape} > 0$ *such that* $\forall e \in \partial K, \forall K \in \mathcal{K}_h, \forall v \in \mathcal{V}_h$

$$\|v\|_{L^2(e)} \leq C_{shape} h_K^{-1/2} \left(\|v\|_{L^2(K)} + h_K |v|_{H^1(K)} \right),$$

where h_K *is the size of the physical element* K *with* C_{shape} *independent of both* K *and* h_k.

Ideas for Proof:

- quasi-uniform mesh.
- requires mappings $\mathbf{G}$ and $\mathbf{\Phi}$.
- assume $\mathcal{O}(H) = 1$.

Properties of the bilinear form $a_{DG}(\cdot, \cdot)$

- DG-norm

$$\|v\|_{DG}^2 = \sum_{i=1}^{N} \alpha_i \|\nabla_\Omega v_i\|_{L^2(\Omega^{(i)})}^2 + \sum_{\Gamma \in \mathcal{E}_I \cup \mathcal{E}_D} \alpha_{(i,j)} \delta^{(i,j)} \int_\Gamma [v]^2.$$

Lemma (Boundedness)

There exists $\overline{\mu}_{DG} > 0$ such that $\forall u, v \in \mathcal{V}_h$

$$|a_{DG}(u, v)| \leq \overline{\mu}_{DG} \|u\|_{DG} \|v\|_{DG},$$

where $\overline{\mu}_{DG}$ is independent of h_i, N and α_i.

Lemma (Coercivity)

There exist $\delta_0 > 0$ and $\underline{\mu}_{DG} > 0$ such that for $\delta > \delta_0$ and $v \in \mathcal{V}_h$

$$a_{DG}(v, v) \geq \underline{\mu}_{DG} \|v\|_{DG}^2,$$

where $\underline{\mu}_{DG}$ is independent of h_i, N and α_i.

Existence and Uniqueness of DG Solution

Theorem

Let $f \in L^2(\Omega), g_D \in H^{\frac{1}{2}}(\Gamma_D)$ and $g_N \in L^2(\Gamma_N)$ be given functions. Then there exists a unique solution $u_h \in \mathcal{V}_h$ such that

$$a_{DG}(u_h, v_h) = \langle F, v_h \rangle, \quad \forall v_h \in \mathcal{V}_h,$$

Moreover, the DG solution u_h continuously depends on the data F.

Proof: follows from Boundedness and Coercivity.

Outline

Lemma (Approximation property)

Assuming quasi-uniformity on each patch. Let κ and s be non-negative real index with $0 < s_i \leq p_i$. We have, for $v \in H^{1+s_i}(\Omega^{(i)})$,

$$|v - \Pi_h v|_{H^{\kappa}(\Omega^{(i)})} \leq C_{shape} h_i^{1+s_i-\kappa} \|v\|_{H^{1+s_i}(\Omega^{(i)})} \tag{4}$$

where $k = 0, 1,\ h_i = \max\{h_k\}_{k \in \mathcal{K}_h^i},\ \forall i = 1, \ldots, N.$

Lemma (DG Approximation property, L+M)

Let $u \in H^{\mathbf{1+s}}(\mathcal{T}_H) \cap H^1(\Omega)$ and $\Pi_h u \in \mathcal{V}_h$ be a continuous interpolant of u. Then, for $s_i > 1/2$, the following bound holds :

$$\|u - \Pi_h u\|_{DG}^2 \leq C_{shape} \sum_{i=1}^{N} \alpha_i h_i^{2s_i} \|u_i\|_{H^{1+s_i}(\Omega^{(i)})}^2 \tag{5}$$

where $\mathbf{s} := (s_1, \ldots, s_N)$ and C_{shape} is independent of h_i, N and the jumps of α_i.

Proof: based on Bazilevs et al.(2006) and Interpolation theory.

Lemma ($\epsilon-$Trace Inequality, L+M)

Let $K \in \mathcal{K}_h, e \in \partial K$. There exists a generic positive constant $C_{shape} > 0$ and $0 < \epsilon < 1/2$ such that $\forall v \in H^{1/2+\epsilon}(\mathcal{T}_H)$

$$\|v\|_{L^2(e)} \leq C_{shape} h_K^{-1/2} \left(\|v\|_{L^2(K)} + h_K^{1/2+\epsilon} |v|_{H^{1/2+\epsilon}(K)} \right),$$

where h_K is the size of the physical element K and C_{shape} independent of K and h_k.

Ideas for Proof:

- quasi-uniform mesh.
- requires mappings $\mathbf{G}$ and $\mathbf{\Phi}$.
- fractional Sobolev space.
- assume $\mathcal{O}(H) = 1$.

DG Approximations for the Surface

Theorem (Error estimate in the DG-norm, L+M)

Let $u \in H^{1+\mathbf{s}}(\mathcal{T}_H)$ with $s_i > 1/2$ be the solution of (1). Let $u_h \in \mathcal{V}_h$ be the solution of (2) and the penalty parameter , δ, be chosen large enough . Then, there exists $D_{shape} > 0$, independent of h_i, N and the jumps of α_i such that the following bound holds:

$$\|u - u_h\|_{DG}^2 \leq C_{shape} \sum_{i=1}^{N} \alpha_i h_i^{2s_i} \|u\|_{H^{1+s_i}(\Omega^{(i)})}^2,$$

where $\mathbf{s} := (s_1, \ldots, s_N) \leq \mathbf{p} := (p_1, \ldots, p_N)$.

Ideas for Proof

- Triangle inequality, Coercivity of $a_{DG}(\cdot, \cdot)$, $\epsilon-$**trace inequality** and Approximation properties.
- Using continuous interpolant $[\Pi_h u] = 0$.

DG Approximations for the Surface

Theorem (Error estimate in the L^2- norm, L+M)

Let $u \in H^{1+s}(\Omega)$ with $s > 1/2$ be the solution of (1) for the Laplace-Beltrami equation ($\alpha = 1$). Let $u_h \in \mathcal{V}_h$ be the solution of (2) and the penalty parameter , δ, be chosen large enough . Then, there exists $E_{shape} > 0$, independent of h_i and N such that the following bound holds:

$$\|u - u_h\|_{L^2(\Omega)} \leq C_{shape} h^{2\mathbf{s}} \left(\sum_{i=1}^{N} \|u_i\|^2_{H^{1+s_i}(\Omega^{(i)})} \right)^{1/2},$$

where $\mathbf{s} := (s_1, \ldots, s_N) \leq \mathbf{p} := (p_1, \ldots, p_N)$ *and* $h := \max\{h_i\}_{i=1}^{N}$.

Ideas for Proof

- Aubin-Nitsche technique, Cauchy Schwarz, Boundedness of $a_{DG}(\cdot, \cdot)$, $\epsilon-$**trace inequality** and Approximation properties.
- Using continuous interpolant $[\Pi_h u] = 0$.
- Using DG-norm estimate.

Outline

Conclusion

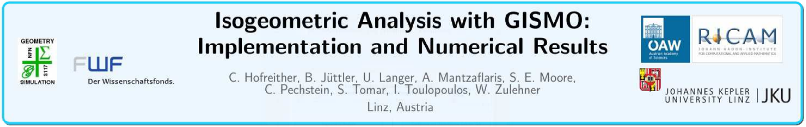
- The **Symmetric IP DG-IGA** method for Diffusion Equations on open and closed surfaces has been established.
- **A-priori error estimates** (DG-norm and L^2-norm) requires estimation of additional terms.
- New **IGA-trace estimate** to prove a-priori error estimates.

Outlook

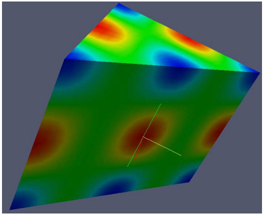
1. Non-matching meshes (similar to matching meshes).
2. **Low-regularity estimates** for $u|_{\Omega^i} \in H^{1+s_i}(\Omega^{(i)})$ with $s_i > 0$ (not $s_i > 1/2$) based on results by
   - Cai, Ye and Zhang, dGfem for interface problems... (SIAM, J. Num. Anal., 2011).
   - Di Pietro and Ern, Analysis of a dGfem for heterogeneous ... (Num. Meth. PDE, 2012).
3. **Solvers** like
   - Kleiss, Jüttler, Pechstein, Tomar, IETI (POSTER).
   - Scacchi, Cho, Pavarino and Beirão da Veiga, BDDC preconditioners for IGA (Math. Models, 2012).
   - Beirão da Veiga, Cho, Pavarino and Scacchi, Overlapping Schwarz methods for IGA (SIAM,J. Num. Anal., 2012).
   - Hofreither and Zulehner Multigrid schemes for IGA (MS-Talk).
   - Beirão da Veiga, Pavarino, Scacchi, Widlund and Zampini, IGA BDDC preconditioners with deluxe scaling (MS-Talk).
4. **Functional type a posteriori estimates** :
   - Kleiss and Tomar, a-posteriori error estimates in IGA (POSTER).

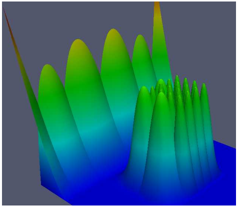
GISMO®

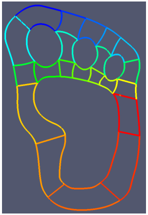
POSTER SESSION



Venue : **Room A** Time : TODAY, **17:00 - 19:15**







THANK YOU!

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