Local simplification of Darcy's equations with pressure dependent permeability

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Ω : bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz-continuous boundary $\partial\Omega$.

We consider a partition of its bounary :

$$\partial\Omega = \overline{\Gamma}_{(p)} \cup \overline{\Gamma}_{(f)} \quad \text{and} \quad \Gamma_{(p)} \cap \Gamma_{(f)} = \emptyset,$$

such that $\partial\Gamma_{(p)}$ and $\partial\Gamma_{(f)}$ are Lipschitz-continuous submanifolds of $\partial\Omega$.

The following nonlinear model was suggested by

K.R. Rajagopal

Where the pressure p presents high variations it is no longer possible to neglect the dependence of the permeability α of the medium with respect to p.

$$
\begin{cases}
\alpha(p)\,\mathbf{u} + \mathbf{grad}\, p = \mathbf{f} & \textbf{in } \Omega, \\
\\
\operatorname{div} \mathbf{u} = 0 & \textbf{in } \Omega, \\
\\
p = p_0 & \textbf{on } \Gamma_{(p)}, \\
\\
\mathbf{u} \cdot \mathbf{n} = g & \textbf{on } \Gamma_{(f)}.
\end{cases}
$$

Unknowns : the velocity $\mathbf{u}$ and the pressure p of the fluid.

Where the pressure p presents high variations it is no longer possible to neglect the dependence of the permeability α of the medium with respect to p. But these variations are negligible in a large part of the domain.

We consider a decomposition of the domain

$$\overline{\Omega} = \overline{\Omega}_\sharp \cup \overline{\Omega}_\flat \quad \textbf{and} \quad \Omega_\sharp \cap \Omega_\flat = \emptyset.$$

$$
\begin{cases}
\alpha(p^*)\,\mathbf{u}^* + \mathbf{grad}\,p^* = \mathbf{f} & \textbf{in } \Omega_\sharp, \\
\alpha_0\,\mathbf{u}^* + \mathbf{grad}\,p^* = \mathbf{f} & \textbf{in } \Omega_\flat, \\
\operatorname{div}\mathbf{u}^* = 0 & \textbf{in } \Omega, \\
p^* = p_0 & \textbf{on } \Gamma_{(p)}, \\
\mathbf{u}^* \cdot \mathbf{n} = g & \textbf{on } \Gamma_{(f)}.
\end{cases}
$$

How to optimize the choice of the decomposition ?

- The full and simplified models
- The discrete problem and its well-posedness
- A posteriori analysis
- Adaptivity strategy
- An iterative algorithm
- A numerical experiment

The full and simplified models

Assume that :

(i) $\Gamma_{(p)}$ **has a positive** $(d-1)$**-measure in** $\partial\Omega$ **;**

(ii) The function α **is a continuous function from** $\mathbb{R}$ **into** $\mathbb{R}$ **and satisfies for two positive constants** α_1 **and** α_2**,**

$$\forall \xi \in \mathbb{R}, \quad \alpha_1 \leq \alpha(\xi) \leq \alpha_2.$$

$$H^1_{(p)}(\Omega) = \left\{ q \in H^1(\Omega);\ q = 0 \ \textbf{ on } \Gamma_{(p)} \right\}.$$

We consider the variational problem :

Find $(\mathbf{u}, p)$ *in* $L^2(\Omega)^d \times H^1(\Omega)$ *such that*

$$p = p_0 \quad \textbf{on } \Gamma_{(p)},$$

and

$$\begin{aligned} &\forall \mathbf{v} \in L^2(\Omega)^d, \quad a^{[p]}(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}, \\ &\forall q \in H^1_{(p)}(\Omega), \quad b(\mathbf{u}, q) = \langle g, q \rangle^{(f)}, \end{aligned}$$

where the bilinear forms $a^{[\xi]}(\cdot,\cdot)$ **for any measurable function** ξ **on** Ω **and** $b(\cdot,\cdot)$ **are defined by**

$$a^{[\xi]}(\mathbf{u}, \mathbf{v}) = \int_\Omega \alpha\Big(\xi(\mathbf{x})\Big)\, \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}, \qquad b(\mathbf{v}, q) = \int_\Omega \mathbf{v}(\mathbf{x}) \cdot (\mathbf{grad}\, q)(\mathbf{x})\, d\mathbf{x}.$$

Here, $\langle \cdot, \cdot \rangle^{(f)}$ **denotes the duality pairing between the dual space** $H^{\frac{1}{2}}_{00}(\Gamma_{(f)})'$ **and** $H^{\frac{1}{2}}_{00}(\Gamma_{(f)})$**.**

Proposition. **Assume that $\mathcal{D}(\Omega \cup \Gamma_{(f)})$ is dense in $H^1_{(p)}(\Omega)$. For any data (f, p_0, g) in $L^2(\Omega)^d \times H^{\frac{1}{2}}(\Gamma_{(p)}) \times H^{\frac{1}{2}}_{00}(\Gamma_{(f)})'$, the full model is equivalent to the previous variational problem, in the sense that any pair (u, p) in $L^2(\Omega)^d \times H^1(\Omega)$ is a solution of the full model in the distribution sense if and only if it is a solution of the variational problem.**

The existence of a solution requires some basic properties of the bilinear forms, first there continuity and also

$$\forall \mathbf{v} \in L^2(\Omega)^d, \qquad a^{[\xi]}(\mathbf{v}, \mathbf{v}) \geq \alpha_1 \|\mathbf{v}\|_{L^2(\Omega)^d},$$

$$\forall q \in H^1_{(p)}(\Omega), \qquad \sup_{\mathbf{v} \in L^2(\Omega)^d} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{L^2(\Omega)^d}} \geq |q|_{H^1(\Omega)}.$$

The existence of a solution is established thanks to Brouwer's fixed point theorem combined with the addition of a penalization term.

Theorem. **For any data** $(\mathbf{f}, p_0, g)$ **in** $L^2(\Omega)^d \times H^{\frac{1}{2}}(\Gamma_{(p)}) \times H_{00}^{\frac{1}{2}}(\Gamma_{(f)})'$, **the variational problem admits a solution** $(\mathbf{u}, p)$ **in** $L^2(\Omega)^d \times H^1(\Omega)$. **Moreover this solution satisfies**

$$\|\mathbf{u}\|_{L^2(\Omega)^d} + \|p\|_{H^1(\Omega)} \leq c\left(\|\mathbf{f}\|_{L^2(\Omega)^d} + \|p_0\|_{H^{\frac{1}{2}}(\Gamma_{(p)})} + \|g\|_{H_{00}^{\frac{1}{2}}(\Gamma_{(f)})'}\right).$$

Assume now that the constant α_0 satisfies

$$\alpha_1 \leq \alpha_0 \leq \alpha_2.$$

We define the function α^* on $\Omega \times I\!\!R$ by

$$\forall \xi \in I\!\!R, \quad \alpha^*(\mathbf{x}, \xi) = \begin{cases} \alpha(\xi) & \textbf{for a.e } \mathbf{x} \textbf{ in } \Omega_\sharp, \\ \alpha_0 & \textbf{for a.e } \mathbf{x} \textbf{ in } \Omega_\flat. \end{cases}$$

A new bilinear form is introduced

$$a^{*[\xi]}(\mathbf{u}, \mathbf{v}) = \int_\Omega \alpha^*\Big(\mathbf{x}, \xi(\mathbf{x})\Big)\, \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}.$$

We consider the “starred" variational problem :

Find $(\mathbf{u}^*, p^*)$ *in* $L^2(\Omega)^d \times H^1(\Omega)$ *such that*

$$p^* = p_0 \quad \textbf{on } \Gamma_{(p)},$$

and

$$\forall \mathbf{v} \in L^2(\Omega)^d, \quad a^{*[p^*]}(\mathbf{u}^*, \mathbf{v}) + b(\mathbf{v}, p^*) = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x},$$
$$\forall q \in H^1_{(p)}(\Omega), \quad b(\mathbf{u}^*, q) = \langle g, q \rangle^{(f)}.$$

Exactly the same arguments as previously lead to

Theorem. **For any data $(\mathbf{f}, p_0, g)$ in $L^2(\Omega)^d \times H^{\frac{1}{2}}(\Gamma_{(p)}) \times H_{00}^{\frac{1}{2}}(\Gamma_{(f)})'$, the starred variational problem admits a solution $(\mathbf{u}^*, p^*)$ in $L^2(\Omega)^d \times H^1(\Omega)$. Moreover this solution satisfies**

$$\|\mathbf{u}^*\|_{L^2(\Omega)^d} + \|p^*\|_{H^1(\Omega)} \leq c \left(\|\mathbf{f}\|_{L^2(\Omega)^d} + \|p_0\|_{H^{\frac{1}{2}}(\Gamma_{(p)})} + \|g\|_{H_{00}^{\frac{1}{2}}(\Gamma_{(f)})'} \right).$$

The links between the solutions $(\mathbf{u}, p)$ and $(\mathbf{u}^*, p^*)$ will be investigated later on.

The discrete problem and its well-posedness

We intend to work with a spectral element discretization.

So, we consider a partition of Ω without overlap into a finite number of rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$) with edges parallel to the coordinate axes :

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \qquad \text{and} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k < k' \leq K.$$

We assume moreover that
(i) both $\overline{\Gamma}_{(p)}$ and $\overline{\Gamma}_{(f)}$ are the union of whole edges ($d = 2$) or faces ($d = 3$) of elements Ω_k,
(ii) the intersection of the boundaries of two subdomains, if not empty, is a vertex, a whole edge or a whole face,
(iii) each Ω_k is contained either in $\Omega_\flat$ or in $\Omega_\sharp$.

The discrete spaces

For each nonnegative integer n, $\mathbb{P}_n(\Omega_k)$ stands for the space of restrictions to Ω_k of polynomials with d variables and degree with respect to each variable $\leq n$.

$$\mathbb{X}_N = \Big\{ v_N \in L^2(\Omega)^d;\ v_N|_{\Omega_k} \in \mathbb{P}_N(\Omega_k)^d,\ 1 \leq k \leq K \Big\},$$
$$\mathbb{M}_N = \Big\{ q_N \in H^1(\Omega);\ q_N|_{\Omega_k} \in \mathbb{P}_N(\Omega_k),\ 1 \leq k \leq K \Big\},$$

and also

$$\mathbb{M}_N^{(p)} = \mathbb{M}_N \cap H^1_{(p)}(\Omega).$$

The quadrature formulas

Gauss–Lobatto formula : There exist a unique set of $N+1$ nodes ξ_j, $0 \le j \le N$, with $\xi_0 = -1$ and $\xi_N = 1$, and a unique set of $N+1$ weights ρ_j, $0 \le j \le N$, such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \rho_j.$$

Denoting by F_k one of the affine mappings that send the square or cube $]-1,1[^d$ onto Ω_k, we define a discrete product on all continuous functions u and v on $\overline{\Omega}_k$ as follows : In dimension $d=2$ for instance

$$(u,v)_N^k = \frac{\mathrm{meas}(\Omega_k)}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} u \circ F_k(\xi_i, \xi_j) v \circ F_k(\xi_i, \xi_j)\, \rho_i \rho_j.$$

This leads to a general discrete product

$$((u,v))_N = \sum_{k=1}^{K} (u,v)_N^k.$$

$\mathcal{I}_N$ **: interpolation operator at all nodes $F_k(\xi_i, \xi_j)$ with values in $\mathbb{M}_N$.**

Similarly, on each edge or face Γ_ℓ of the Ω_k, assuming for instance that the mapping F_k maps $\{-1\}\times]-1,1[^{d-1}$ onto Γ_ℓ, we define a discrete product : In dimension $d = 2$ for instance,

$$(u, v)_N^{\Gamma_\ell} = \frac{\text{meas}(\Gamma_\ell)}{2} \sum_{j=0}^{N} u \circ F_k(\xi_0, \xi_j) v \circ F_k(\xi_0, \xi_j)\, \rho_j.$$

A global product on $\Gamma_{(f)}$ is then defined by

$$((u, v))_N^{(f)} = \sum_{\ell \in \mathcal{L}_{(f)}} (u, v)_N^{\Gamma_\ell},$$

where $\mathcal{L}_{(f)}$ stands for the set of indices ℓ such that Γ_ℓ is contained in $\Gamma_{(f)}$.

Finally, assuming that p_0 is continuous on $\overline{\Gamma}_{(p)}$, for each edge ($d=2$) or face ($d=3$) Γ_ℓ of an element Ω_k which is contained in $\Gamma_{(p)}$, $p_{0N}|_{\Gamma_\ell}$ belongs to $\mathbb{P}_N(\Gamma_\ell)$ and is equal to p_0 at the $(N+1)^{d-1}$ nodes $F_k(\xi_i, \xi_j)$ or $F_k(\xi_i, \xi_j, \xi_m)$ which are located on $\overline{\Gamma}_\ell$.

We denote by $i_N^{(p)}$ the corresponding interpolation operator.

We assume that all data f, p_0 and g are continuous where needed. The discrete problem reads

Find $(\mathbf{u}_N, p_N)$ *in* $\mathbb{X}_N \times \mathbb{M}_N$ *such that*

$$p_N = p_{0N} \quad \textbf{on } \Gamma_{(p)},$$

and

$$\begin{aligned} &\forall \mathbf{v}_N \in \mathbb{X}_N, \quad a_N^{*[p_N]}(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N) = ((\mathbf{f}, \mathbf{v}_N))_N, \\ &\forall q_N \in \mathbb{M}_N^{(p)}, \quad b_N(\mathbf{u}_N, q_N) = ((g, q_N))_N^{(f)}, \end{aligned}$$

where the bilinear forms $a_N^{*[\xi]}(\cdot,\cdot)$ for any continuous function ξ on $\overline{\Omega}$ and $b_N(\cdot,\cdot)$ are defined by

$$a_N^{*[\xi]}(\mathbf{u}, \mathbf{v}) = ((\alpha^*(\cdot,\xi)\,\mathbf{u}, \mathbf{v}))_N, \qquad b_N(\mathbf{v}, q) = ((\mathbf{v}, \mathbf{grad}\, q))_N.$$

As now standard, the well-posedness of this problem and a priori error estimates are now deduced from the theorem due to

F. Brezzi, J. Rappaz, P.-A. Raviart

This approach requires the stability and optimal a priori error estimates for the linear problem (i.e., when $\Omega_\flat = \Omega$) which are known for a long time.

Theorem, Part I. **Assume that**

(i) **the coefficient** α **is of class** $\mathcal{C}^2$ **on** $\mathbb{R}$ **with bounded derivatives ;**

(ii) **the solution** $U^* = (\mathrm{u}^*, p^*)$ **of the simplified problem belongs to** $H^s(\Omega)^d \times H^{s+1}(\Omega)$ **for some** $s > 0$ **in dimension** $d = 2$ **and** $s > 1$ **in dimension** $d = 3$ **;**

(iii) **the solution** $U^* = (\mathrm{u}^*, p^*)$ **of the simplified problem is nonsingular ;**

(iv) **the data** (f, p_0, g) **belong to** $H^\sigma(\Omega)^d \times H^{\sigma+\frac{1}{2}}(\Gamma_{(p)}) \times H^\sigma(\Gamma_{(f)})$**,** $\sigma > \frac{d}{2}$**.**

There exist a positive integer N^* **and a positive constant** ρ **such that, for** $N \geq N^*$**, the discrete problem has a unique solution** (u_N, p_N) **in the ball with centre** (u^*, p^*) **and radius** $\rho\,\mu(N)^{-1}$**, with** $\mu(N)$ **equal to** $|\log N|^{\frac{1}{2}}$ **in dimension** $d = 2$ **and to** N **in dimension** $d = 3$**.**

Theorem, Part II. **Moreover this solution satisfies the following a priori error estimate**

$$\begin{aligned}
&\|\mathbf{u}^* - \mathbf{u}_N\|_{L^2(\Omega)^d} + \|p^* - p_N\|_{H^1(\Omega)} \\
&\qquad \le c(\mathbf{u}^*, p^*) \Big(N^{-s} \left(\|\mathbf{u}^*\|_{H^s(\Omega)^d} + \|p^*\|_{H^{s+1}(\Omega)} \right) \\
&\qquad\qquad + N^{-\sigma} \left(\|\mathbf{f}\|_{H^\sigma(\Omega)^d} + \|p_0\|_{H^{\sigma+\frac{1}{2}}(\Gamma_{(p)})} + \|g\|_{H^\sigma(\Gamma_{(f)})} \right) \Big),
\end{aligned}$$

where the constant $c(\mathbf{u}^*, p^*)$ only depends on the solution $(\mathbf{u}^*, p^*)$.

A posteriori analysis

As now standard for multistep discretizations, the a posteriori analysis that we perform relies on the triangle inequalities

$$\|\mathbf{u}-\mathbf{u}_N\|_{L^2(\Omega)^d} \leq \|\mathbf{u}-\mathbf{u}^*\|_{L^2(\Omega)^d} + \|\mathbf{u}^*-\mathbf{u}_N\|_{L^2(\Omega)^d},$$

$$\|p-p_N\|_{H^1(\Omega)} \leq \|p-p^*\|_{H^1(\Omega)} + \|p^*-p_N\|_{H^1(\Omega)}.$$

Indeed, we wish to uncouple as much as possible the errors issued from the simplification and the discretization.

Error due to the simplification of the model

On each domain Ω_k, $1 \leq k \leq K$, the error indicator is defined by

$$\eta_{N,k}^{(s)} = \|\Big(\alpha(p_N) - \alpha^*(\cdot, p_N)\Big)\mathbf{u}_N\|_{L^2(\Omega_k)^d}.$$

It can be noted that all $\eta_{N,k}^{(s)}$ such that Ω_k is contained in $\Omega_\sharp$ are zero. Otherwise, they are given by

$$\eta_{N,k}^{(s)} = \|\Big(\alpha(p_N) - \alpha_0\Big)\mathbf{u}_N\|_{L^2(\Omega_k)^d}.$$

In all cases, computing them is easy.

Proposition. **If the solution $U=(\mathbf{u},p)$ of the continuous problem**
(i) belongs to $H^s(\Omega)^d \times H^{s+1}(\Omega)$ for some $s>0$ in dimension $d=2$ and $s>\frac{1}{2}$ in dimension $d=3$;
(ii) is nonsingular,
there exists a neighbourhood of U in $H^s(\Omega)^d \times H^{s+1}(\Omega)$ such that the following a posteriori error estimate holds for any solution $U^*=(\mathbf{u}^*,p^*)$ of the simplified problem in this neighbourhood

$$\|\mathbf{u}-\mathbf{u}^*\|_{L^2(\Omega)^d}+\|p-p^*\|_{H^1(\Omega)} \leq c(\mathbf{u},p)\Big(\Big(\sum_{k=1}^{K}(\eta_{N,k}^{(s)})^2\Big)^{\frac{1}{2}}+\|\mathbf{u}^*-\mathbf{u}_N\|_{L^2(\Omega)^d}+\|p^*-p_N\|_{H^1(\Omega)}\Big),$$

where the constant $c(\mathbf{u},p)$ only depends on the solution U.

First estimate of the error due to the simplification !

The residual equation can be written explicitly. It reads

$$
\begin{aligned}
\forall \mathbf{v} \in L^2(\Omega)^d, \quad & a^{[p]}(\mathbf{u}-\mathbf{u}^*, \mathbf{v}) + b(\mathbf{v}, p - p^*) \\
& \qquad = - \int_\Omega \Big(\alpha(p) - \alpha^*(\mathbf{x}, p^*)\Big)\mathbf{u}^*(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}, \\
\forall q \in H^1_{(p)}(\Omega), \quad & b(\mathbf{u}-\mathbf{u}^*, q) = 0.
\end{aligned}
$$

This leads to the next result.

Proposition. If the previous assumptions hold, the following estimate holds for each indicator $\eta^{(s)}_{N,k}$

$$
\eta^{(s)}_{N,k} \le c\left(\|\mathbf{u}-\mathbf{u}^*\|_{L^2(\Omega_k)^d} + \|p-p^*\|_{H^1(\Omega_k)} + \|\mathbf{u}^*-\mathbf{u}_N\|_{L^2(\Omega_k)^d} + \|p^*-p_N\|_{H^1(\Omega_k)}\right).
$$

Error due to the discretization

Some further notation : For $1 \leq k \leq K$, let $\mathcal{E}_k^0$ and $\mathcal{E}_k^{(f)}$ be the set of edges ($d = 2$) or faces ($d = 3$) of Ω_k which are not contained in $\partial\Omega$ or are contained in $\overline{\Gamma}_{(f)}$, respectively.

We also introduce an approximation g_N of g : Assuming that g is continuous on $\overline{\Gamma}_{(f)}$, for each edge ($d = 2$) or face ($d = 3$) Γ_ℓ of an element Ω_k which is contained in $\Gamma_{(f)}$, $g_N|_{\Gamma_\ell}$ belongs to $\mathbb{P}_N(\Gamma_\ell)$ and is equal to g at the $(N+1)^{d-1}$ nodes $F_k(\xi_i, \xi_j)$ or $F_k(\xi_i, \xi_j, \xi_m)$ which are located on $\overline{\Gamma}_\ell$.

On each domain Ω_k, $1 \leq k \leq K$, the error indicator is defined by

$$\eta_{N,k}^{(d)} = \|\mathcal{I}_N \mathbf{f} - \alpha^*(\cdot, p_N)\,\mathbf{u}_N - \mathbf{grad}\,p_N\|_{L^2(\Omega_k)^d} + N^{-1}\,\|\mathrm{div}\,\mathbf{u}_N\|_{L^2(\Omega_k)}$$
$$+ \sum_{\gamma \in \mathcal{E}_k^0} N^{-\frac{1}{2}}\,\|[\mathbf{u}_N \cdot \mathbf{n}]_\gamma\|_{L^2(\gamma)} + \sum_{\gamma \in \mathcal{E}_k^{(f)}} N^{-\frac{1}{2}}\,\|g_N - \mathbf{u}_N \cdot \mathbf{n}\|_{L^2(\gamma)}.$$

The residual equations read, for all $\mathbf{v}$ in $L^2(\Omega)^d$,

$$
\begin{aligned}
&a^{*[p^*]}(\mathbf{u}^*, \mathbf{v}) - a^{*[p_N]}(\mathbf{u}_N, \mathbf{v}) + b(\mathbf{v}, p^* - p_N) \\
&\quad = \int_\Omega \Big(\mathcal{I}_N \mathbf{f} - \alpha^*(\mathbf{x}, p_N)\, \mathbf{u}_N - \mathbf{grad}\, p_N \Big)(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, \underset{\cdot}{\mathbf{x}} + \int_\Omega \big(\mathbf{f} - \mathcal{I}_N \mathbf{f} \big)(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, \underset{\cdot}{\mathbf{x}},
\end{aligned}
$$

and, for all q in $H^1_{(p)}(\Omega)$,

$$
b(\mathbf{u}^* - \mathbf{u}_N, q) = \langle g, q \rangle^{(f)} - b(\mathbf{u}_N, q).
$$

A further integration by parts is necessary to handle this last equation

$$
\begin{aligned}
&b(\mathbf{u}^* - \mathbf{u}_N, q) = \langle g - g_N, q \rangle^{(f)} + \langle g_N, q - q_N \rangle^{(f)} \\
&\qquad + \sum_{k=1}^{K} \Big(\int_{\Omega_k} (\mathrm{div}\, \mathbf{u}_N)(\mathbf{x})(q - q_N)(\mathbf{x})\, d\mathbf{x} - \int_{\partial\Omega_k} (\mathbf{u}_N \cdot \mathbf{n})(\tau)(q - q_N)(\tau)\, d\tau \Big).
\end{aligned}
$$

Let $\rho(\Omega)$ be equal to 1 if the domain Ω is either two-dimensional or convex, to $N^{\frac{1}{2}}$ otherwise.

J. Pousin, J. Rappaz

Proposition. If the solution $U^* = (\mathbf{u}^*, p^*)$ of the simplified problem
(i) belongs to $H^s(\Omega)^d \times H^{s+1}(\Omega)$ for some $s > 0$ in dimension $d = 2$ and $s > \frac{1}{2}$ in dimension $d = 3$;
(ii) is nonsingular,
there exists a neighbourhood of U^* such that the following a posteriori error estimate holds for any solution $U_N = (\mathbf{u}_N, p_N)$ of the discrete problem in this neighbourhood

$$\|\mathbf{u}^* - \mathbf{u}_N\|_{L^2(\Omega)^d} + \|p^* - p_N\|_{H^1(\Omega)} \leq c(\mathbf{u}^*, p^*) \Big(\rho(\Omega) \Big(\sum_{k=1}^{K} (\eta_{N,k}^{(d)})^2\Big)^{\frac{1}{2}}$$
$$+ \|\mathbf{f} - \mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d} + \|p_0 - p_{0N}\|_{H^{\frac{1}{2}}(\Gamma_{(p)})} + \|g - g_N\|_{H^{\frac{1}{2}}_{00}(\Gamma_{(f)})'}\Big),$$

where the constant $c(\mathbf{u}^*, p^*)$ only depends on the solution U^*.

Summary of the results

Up to the terms involving the data, namely

$$\|\mathbf{f}-\mathcal{I}_N\mathbf{f}\|_{L^2(\Omega)^d}+\|p_0-p_{0N}\|_{H^{\frac{1}{2}}(\Gamma_{(p)})}+\|g-g_N\|_{H^{\frac{1}{2}}_{00}(\Gamma_{(f)})'}$$

the full error

$$E=\|\mathbf{u}-\mathbf{u}^*\|_{L^2(\Omega)^d}+\|p-p^*\|_{H^1(\Omega)}+\|\mathbf{u}^*-\mathbf{u}_N\|_{L^2(\Omega)^d}+\|p^*-p_N\|_{H^1(\Omega)},$$

satisfies

$$E\le c\Big(\sum_{k=1}^{K}\big((\eta_{N,k}^{(s)})^2+\rho(\Omega)^2(\eta_{N,k}^{(d)})^2\big)\Big)^{\frac{1}{2}}.$$

This estimate is fully optimal when the domain Ω is two-dimensional or convex. Moreover, for three-dimensional non-convex domains Ω, the lack of optimality only concerns the terms $\|\operatorname{div}\mathbf{u}_N\|_{L^2(\Omega_k)}$.

The indicators $\eta_{N,k}^{(s)}$ seem to form an efficient tool for the automatic simplification of the model, as described in the following strategy.

Adaptivity strategy

Let η^* be a fixed tolerance.

From now on, we work with N sufficiently large for the quantities linked to the data to be smaller than η^*.

Initialization step. **We first work with the partition of Ω given by**

$$\Omega_\sharp^0 = \emptyset, \qquad \Omega_\flat^0 = \Omega,$$

and we solve the corresponding linear problem.

Adaptation step. **Assuming that a partition of Ω into $\Omega_\sharp^m$ and $\Omega_\flat^m$ is given, we compute the corresponding solution (u_N, p_N) of the discrete problem, the indicators $\eta_{N,k}^{(s)}$ and their mean value $\overline{\eta}_N^{(s)}$, the indicators $\eta_{N,k}^{(d)}$ and their mean value $\overline{\eta}_N^{(d)}$. The new partition of Ω is thus constructed in the following way :**

(i) The domain $\Omega_\sharp^{m+1}$ is the union of $\Omega_\sharp^m$ and of all Ω_k such that

$$\eta_{N,k}^{(s)} \geq \max\,\{\overline{\eta}_N^{(s)}, \overline{\eta}_N^{(d)}\};$$

(ii) The domain $\Omega_\flat^{m+1}$ is taken equal to $\Omega \setminus \overline{\Omega}_\sharp^{m+1}$.

The adaptation step must be iterated either a fixed number of times or until the Hilbertian sum $\left(\sum_{k=1}^{K}(\eta_{N,k}^{(s)})^2\right)^{\frac{1}{2}}$ **becomes smaller than** η^* **(when possible).**

There is no proof of convergence of the partition of Ω **into** $\Omega_\sharp^m$ **and** $\Omega_\flat^m$**.**

An iterative algorithm

Assuming that an initial guess (u_N^0, p_N^0) is given, we solve iteratively the problems

Find $(\mathbf{u}_N^n, p_N^n)$ in $\mathbb{X}_N \times \mathbb{M}_N$ such that

$$p_N^n = p_{0N} \quad \textbf{on } \Gamma_{(p)},$$

and

$$\begin{aligned} &\forall \mathbf{v}_N \in \mathbb{X}_N, \quad a_N^{*[p_N^{n-1}]}(\mathbf{u}_N^n, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N^n) = ((\mathbf{f}, \mathbf{v}_N))_N, \\ &\forall q_N \in \mathbb{M}_N^{(p)}, \quad b_N(\mathbf{u}_N^n, q_N) = ((g, q_N))_N^{(f)}. \end{aligned}$$

It is readiy checked that there exists a constant λ only depending on U^* such that any solution $(\mathbf{u}_N, p_N)$ of the discrete problem satisfies

$$\|\mathbf{u}_N\|_{L^\rho(\Omega)^d} \leq \lambda,$$

with $\rho > 2$ in dimension $d = 2$ and $\rho = 3$ in dimension $d = 3$.

Proposition. When all previous assumptions hold, there exists a positive constant c_0 independent of N such that, if

$$\lambda\alpha^\dagger(1 + \frac{\alpha_2}{\alpha_1}) < c_0,$$

the sequence $(\mathbf{u}_N^n, p_N^n)_n$ converges to $(\mathbf{u}_N, p_N)$ in $H^1(\Omega)^d \times L^2(\Omega)$. Moreover, the following estimate holds with $\kappa = \lambda\alpha^\dagger(1 + \frac{\alpha_2}{\alpha_1})\, c_0^{-1}$,

$$\|\mathbf{u}_N - \mathbf{u}_N^n\|_{L^2(\Omega)^d} \leq c\,\frac{\lambda\alpha^\dagger}{\alpha_1}\,\kappa^{n-1}, \qquad \|p_N - p_N^n\|_{H^1(\Omega)} \leq \kappa^n.$$

A posteriori analysis

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In each domain Ω_k, $1 \leq k \leq K$, we define the error indicator

$$\eta_{N,k,n}^{(ia)} = \|\mathcal{I}_N\Big(\alpha^*(\cdot, p_N^n) - \alpha^*(\cdot, p_N^{n-1})\Big)\mathbf{u}_N^n\|_{L^2(\Omega_k)^d}.$$

Here also, all $\eta_{N,k,n}^{(ia)}$ such that Ω_k is contained in $\Omega_\flat$ are zero.

Proposition. When all previous assumptions holds, there exists a constant ν such that the following a posteriori error estimate holds for any solution $U_N^n = (\mathbf{u}_N^n, p_N^n)$ in the ball with centre U_N and radius $\nu\mu(N)^{-1}$,

$$\|\mathbf{u}_N - \mathbf{u}_N^n\|_{L^2(\Omega)^d} + \|p_N - p_N^n\|_{H^1(\Omega)} \leq c\,\Big(\sum_{k=1}^{K} (\eta_{N,k,n}^{(ia)})^2\Big)^{\frac{1}{2}}.$$

where the constant c is independent of N.

An upper bound for each $\eta_{N,k,n}^{(ia)}$ can also be proven.

The error indicators provide the appropriate tool for stopping the iterative algoritthm at the right step. Moreover this algorithm can be applied on $\Omega_\flat$ only one step over ? ?.

A numerical experiment

We work on the domain

$$\Omega =]-1,1[^2, \qquad \Gamma_{(p)} = \{-1\} \times]-1,1[, \qquad \Gamma_{(f)} = \partial\Omega \setminus \overline{\Gamma}_{(p)}.$$

The function α is equal to

$$\alpha(\xi) = \exp(\xi),$$

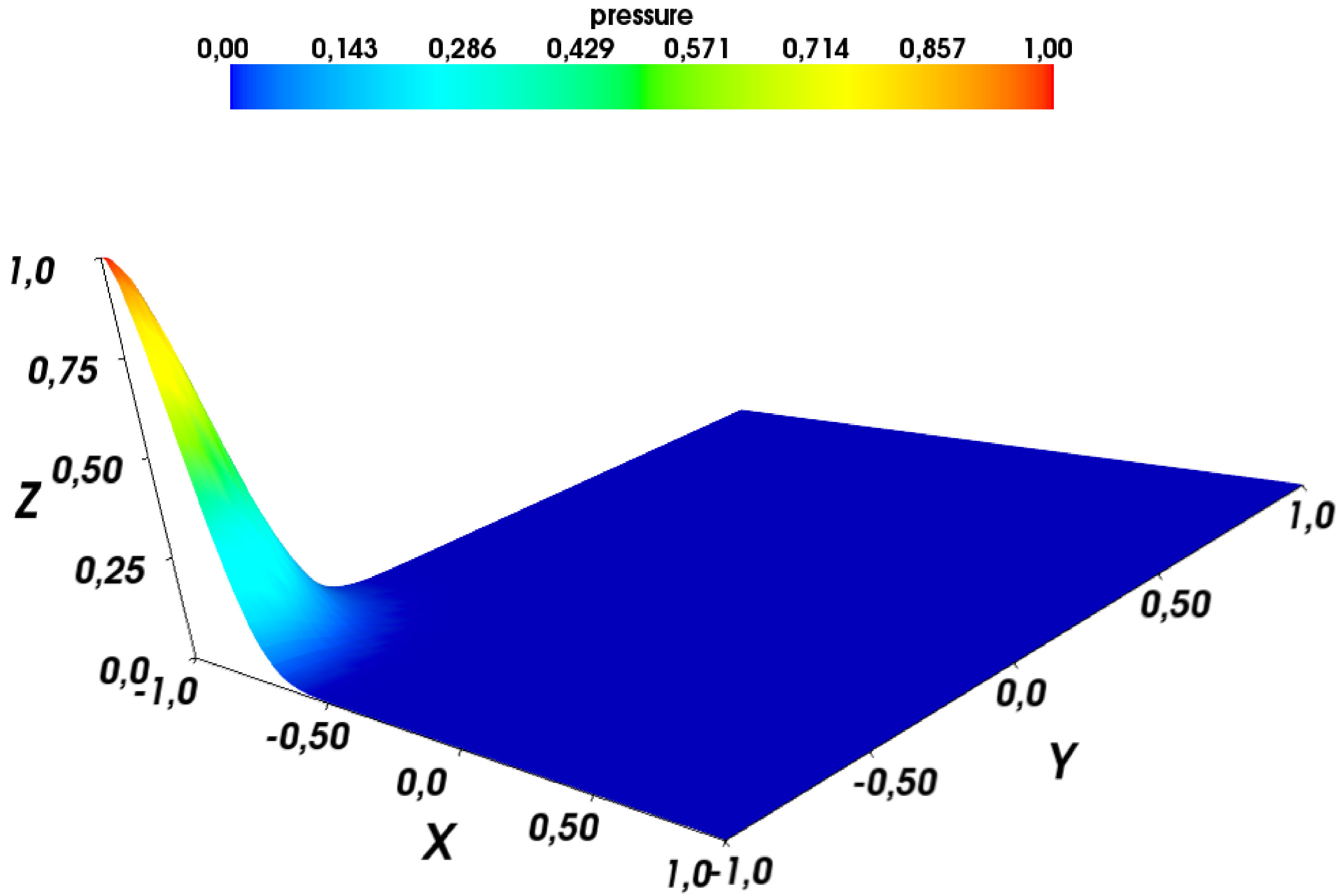
truncated at $\alpha_1 = \frac{3}{4}$ and $\alpha_2 = 3$.

We consider the given solution

$$\mathbf{u}(x,y) = \Big(\sin(x)\,\cos(y), -\cos(x)\,\sin(y)\Big),$$

$$p(x,y) = \exp\Big(-\frac{(x+1)^2 + (y+1)^2}{0.05}\Big).$$

The fact that the pressure presents high variations only on a part of the domain seems well appropriate for studying a possible simplification of the problem.



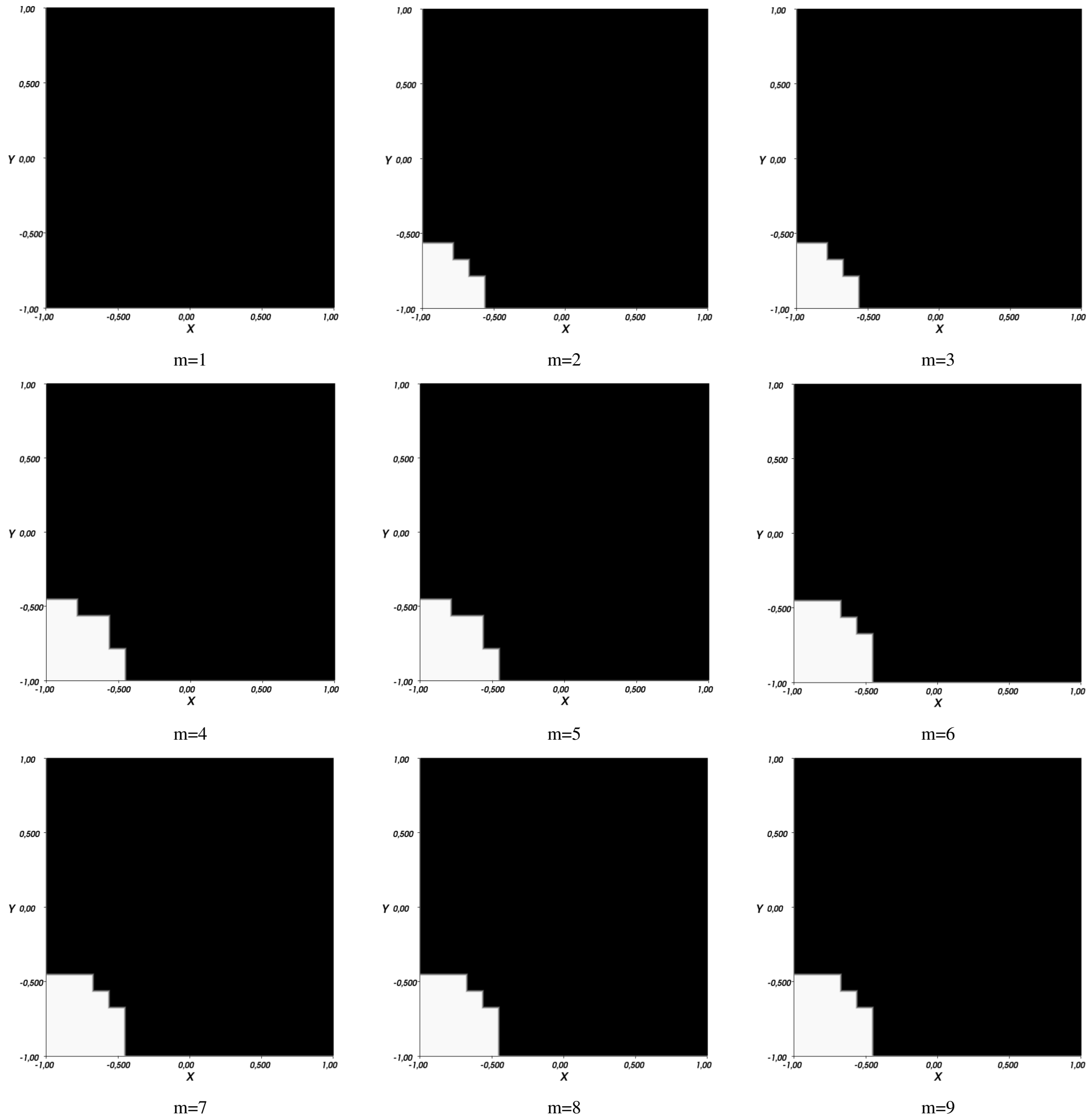
The pressure

The discretization is performed with low degree polynomials : $N = 4$ but many elements : $K = 324 = 18^2$ equal squares.

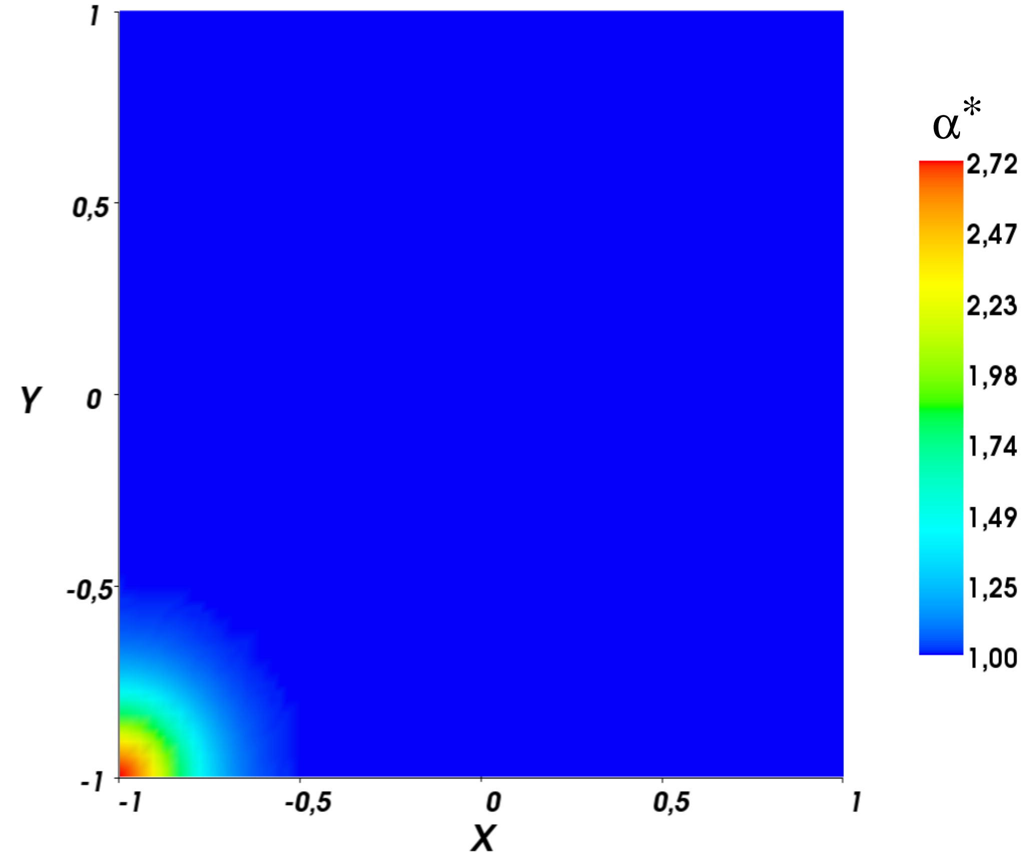
We follow the previous adaptivity strategy procedure with $\eta^* = 10^{-8}$.

The convergence is obtained for $m = 9$, which proves the efficiency of our strategy.

It can be noted that $\Omega_\sharp^9$ contains 22 elements.



The successive partitions of Ω into $\Omega_\sharp$ and $\Omega_\flat$



The isovalues of the final function α^*

Interest of the simplification

The iterative algorithm is performed as follows : Each iteration is applied on $\Omega_\sharp$ and only one iteration over 4 is applied on the whole domain.

	Without simplification	**With simplification**
Number of iterations	**7**	9
CPU time(s)	4.32	1.06

Comparison of the discretizations with and without simplification

Thank you for your attention