An Optimal Preconditioner for Parallel Adaptive Finite Elements

Sébastien Loisel & Hieu Nguyen

Department of Mathematics
Heriot-Watt University

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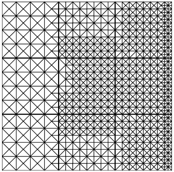
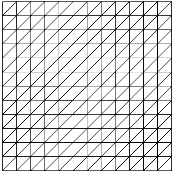
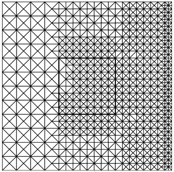
DDMs and AFEs



DDMs:

- start with a fine mesh
- prefer independence of work associated with each subdomain



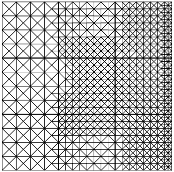


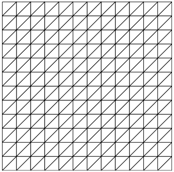
AFEs:

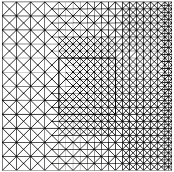
- build meshes gradually
- Global information (computed solutions, error estimates, mesh status) is usually needed.

How can we combine these very different methods?

DDMs and AFEs







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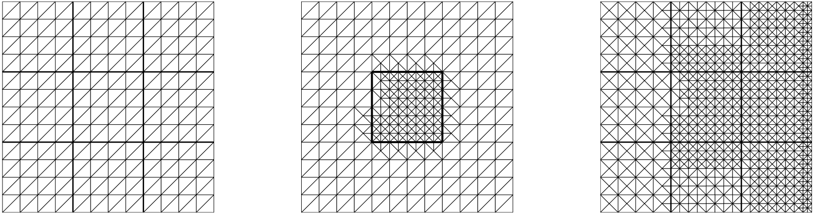
A Parallel Adaptive Meshing Algorithm



- Step I - Initialization: A coarse mesh is partitioned into subdomains.
- Step II - Adaptive Enrichment: Each processor gets complete coarse mesh. Each processor independently solves the entire problem but adaptively focus the adaptive enrichment on its subdomain. Regularize local meshes so that the global mesh is conforming
- Step III - DD Solve: Compute global solution using a DD solver.

A brief history

- The parallel adaptive meshing was first introduced in [Bank & Holst 2000, 2003]
- AFEs library `deal.ii` adopts the approach and show scalability up to 16384 processors: [Bangerth et al. 2011]

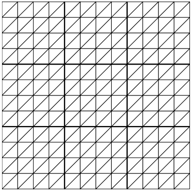
DD Solver for this type of parallel algorithm: [Bank& Lu 2004]

- use DDM as a solver
- final system is non-symmetric
- convergence result is limited

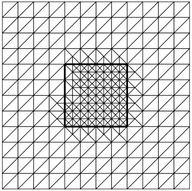
Model Problem and Notations

Find $u \in H_0^1(\Omega)$ such that

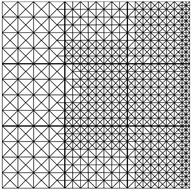
$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \qquad \text{on } \partial\Omega \end{aligned} .$$



$\mathcal{T}_H,\ V_0,\ \psi_j^{(0)}(x)$



$\mathcal{T}_i,\ V_i,\ \psi_j^{(i)}(x)$



$\mathcal{T}_h,\ V_h,\ \psi_j(x)$

- Partition: $\bar{\Omega} = \cup_{i=1}^{N} \bar{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$
- Meshes: $\mathcal{T}_H \subset \mathcal{T}_i \subset \mathcal{T}_h$, and $\mathbb{P}_1$ FE spaces: $V_H \equiv V_0 \subset V_i \subset V_h$
- Nodal basis functions: $\{\psi_j^{(0)}\}_1^{n_0}$, $\{\psi_j^{(i)}\}_1^{n_i}$, $\{\psi_j\}_1^{n}$
- Linear system of the approximated solution: $Au = f$.

Preconditioner Formulation

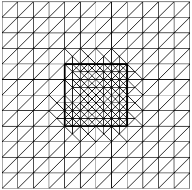
Preconditioner and preconditioned system:

$$P^{-1} = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i,$$

$$P^{-1}A = \sum_{i=1}^{N} P_i = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i A,$$

where $R_i^T \in \mathbb{R}^{n \times n_i}$ is the extension matrix from V_i to V_h, $A_i = R_i A R_i^T$.

- No explicit coarse component
- A_i can be assembled locally
- P^{-1} is symmetric
- P_i is an "A-orthogonal projection onto V_i"



Spectrum Study

Lemma 1

There exist an A-orthogonal matrix U, Euclidean projections Q_i and $\hat{Q}_i$ such that:

$$P_i = UQ_iU^{-1} = U \begin{bmatrix} I & 0 \\ 0 & \hat{Q}_i \end{bmatrix} U^{-1},$$

$$P^{-1}A = \sum_{i=1}^{N} P_i \sim Q_i = \begin{bmatrix} NI & 0 \\ 0 & \sum_{i=1}^{N} \hat{Q}_i \end{bmatrix}.$$

In addition,

$$\sigma(P^{-1}A) \subset [\hat{\lambda}_{\min}, \hat{\lambda}_{\max}] \cup \{N\},$$

where $\hat{\lambda}_{\min}$ and $\hat{\lambda}_{\max}$ are the smallest and largest eigenvalues of $\sum_{i=1}^{N} \hat{Q}_i$ and $0 < \hat{\lambda}_{\min} \leq \hat{\lambda}_{\max} \leq N$.

Convergence of CG and GMRES

Theorem 2

The errors e_k of the CG method and the residuals r_k of the GMRES when solving the system $Au = f$ with left-preconditioner P^{-1} satisfy

$$\frac{\|e_k\|_A}{\|e_0\|_A} \leq \frac{2(N - \hat{\lambda}_{\min})}{N} \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^{k-1} < 2 \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^{k-1},$$

$$\frac{\|r_k\|}{\|f\|} \leq 2\sqrt{\kappa(A)} \, \frac{(N - \hat{\lambda}_{\min})}{N} \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^{k-1} < 2\sqrt{\kappa(A)} \left(\frac{\sqrt{\hat{\kappa}} - 1}{\sqrt{\hat{\kappa}} + 1} \right)^{k-1},$$

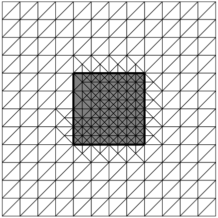
where $\hat{\kappa} = \hat{\lambda}_{\max}/\hat{\lambda}_{\min}$ is called the effective condition number of $P^{-1}A$.

Proof: Consider $q(x) = \dfrac{T_{k-1}(\gamma - \frac{2x}{\hat{\lambda}_{\max} - \hat{\lambda}_{\min}})(N - x)}{N T_{k-1}(\gamma)}$.

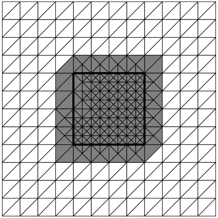
Now we need to estimate $\hat{\lambda}_{\max}$ and $\hat{\lambda}_{\min}$, the second-largest and smallest eigenvalues of $P^{-1}A$.

Some Assumptions

Each Ω_i is extended to $\tilde{\Omega}_i$ by adding layers of elements in $\mathcal{T}_i$:



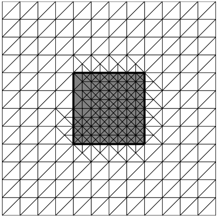
Ω_i (shaded area)



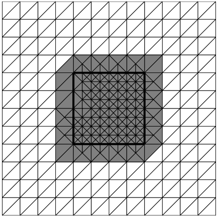
$\tilde{\Omega}_i$ (shaded area)

- $\mathcal{T}_i|_{\tilde{\Omega}_i^c} \equiv \mathcal{T}_H|_{\tilde{\Omega}_i^c}$
- $d(\tilde{\Omega}_i \backslash \Omega_i) = O(H)$
- $\{\tilde{\Omega}_i\}_{i=1}^N$ can be colored using N^c colors.

Cut-off functions



Ω_i (shaded area)



$\tilde{\Omega}_i$ (shaded area)

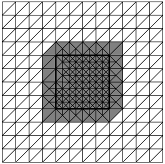
For each Ω_i of color c_k, we define $\theta_i^{(c_k)}(x)$ such that

- $\mathrm{supp}(\theta_i^{(c_k)}) \subset \bar{\tilde{\Omega}}_i$,
- $0 \le \theta_i^{(c_k)}(x) \le 1, \quad$ for all $x \in \bar{\tilde{\Omega}}_i$,
- $\theta_i^{(c_k)}(x) = 1, \quad$ for all $x \in \bar{\Omega}_i$,
- $\|\nabla \theta_i^{(c_k)}\|_\infty \le C^\theta / H$, where C^θ does not depend on i and H.

Strengthened Cauchy Inequality

Let $V_i^\dagger = \mathrm{span}\{\psi_j^{(i)} | \psi_j^{(i)} \notin V_0\}$.

- $V_0 \oplus V_i^\dagger = V_i$.
- $V_i^\dagger \subset \tilde{V}_i = V_h \cap H_0^1(\tilde{\Omega}_i)$



Lemma 3 ([Bank 96])

For $v_0(x) \in V_0$ and $v_i^\dagger(x) \in V_i^\dagger$, there exist $0 < \gamma < 1$ such that

$$|a(v_0, v_i^\dagger)| \leq \gamma \|v_0\|_A \|v_i^\dagger\|_A.$$

where γ depends on the PDE, the shape regularity quality of $\mathcal{T}_H$, $\mathcal{T}_i$, but is otherwise independent of the mesh sizes h and H.

Second Largest Eigenvalue Estimate

Theorem 4

The second largest eigenvalue of the preconditioned system $P^{-1}A$

$$\hat{\lambda}_{\max} \leq \frac{N^c}{(1-\gamma^2)}.$$

Sketch of Proof: Assume $V_i^{\dagger}$ is spaned by (A-orthogonal) columns of W_i and let $F_i = U^{-1}W_i = [X_i^T \; Y_i^T]^T$.

(i) $Q_i = \begin{bmatrix} I & 0 \\ 0 & Y_i(Y_i^TY_i)^{-1}Y_i^T \end{bmatrix}$, or $\hat{Q}_i = Y_i(Y_i^TY_i)^{-1}Y_i^T$.

(ii) $(1-\gamma^2)I \preceq Y_i^TY_i$, ($\preceq$ denotes the positive semi-definition ordering)

(iii) $\sum_{i=1}^N Y_iY_i^T \preceq \sum_{i=1}^N \tilde{Y}_i\tilde{Y}_i^T$.vs $\sum_{i=1}^N \tilde{F}_i\tilde{F}_i^T = \sum_{i=1}^N \tilde{P}_i \preceq N^c \, I_{n-n_0}$.

Smallest Eigenvalue Estimate

Theorem 5

For $u(x) \in V_h$ there exists $u_i(x) \in V_i$ s.t $u = \sum_{i=1}^{N} u_i$ and

$$\sum_{i=1}^{N} a(u_i, u_i) \leq C_m \, a(u, u), \quad (\hat{\lambda}_{\min} \geq C_m^{-1})$$

where C_m is a constant independent of H, h and N. In addition, if

$$C^I \geq 1, \; C^\theta \geq 1, \; N^n \geq 8,$$

then

$$\hat{\lambda}_{\min} \geq \left(\frac{83}{45} \left(\frac{45}{4} (N^n)^2 (C^I)^4 (C^\theta)^2 \right)^{N^c} \right)^{-1}.$$

Estimate $\hat{\lambda}_{\min}$: Interpolation Operators

Given a mesh $\mathcal{T}^\circ$, choose for each node $x_j^\circ \in \mathcal{T}^\circ$ an edge $e^\circ \ni x_j^\circ$, define $I^\circ = I_{\mathcal{T}^\circ}^{\{e_j^\circ\}} : H^1(\Omega) \to V^\circ$, based on [Scott & Zhang 90]:

$$I^\circ u(x) = \sum_{j=1}^{n_i} \psi_j^\circ(x) \int_{e_j^\circ} \eta_j^\circ(\xi) u(\xi)\, d\xi,$$

where η_j° is $L^2(e_j^\circ)$-dual basis functions $\int_{e_j^\circ} \eta_j^\circ\, \psi_k^\circ = \delta_{jk},\ k = 1, \dots, n^\circ$.

- Need a systematic way to choose edges $\{e_j^\circ\}$
- Stability properties $\quad (I_i^{h,H} = I_{V_0}^{\{e_j^{(i)}\}},\ I_i^H = I_{V_0}^{\{e_j^{(0)}\}}\ )$

$$\begin{aligned}
\|I_i^{h,H} u\|_{H^1(K)} &\le C^I\ |u|_{H^1(w_K)}, && K, w_K \in \mathcal{T}_i,\\
\|u - I^H u\|_{L^2(K)} &\le C^I\ H |u|_{H^1(w_K)}, && K, w_K \in \mathcal{T}_H,\\
\|I^H u\|_{H^1(K)} &\le C^I\ |u|_{H^1(w_K)}, && K, w_K \in \mathcal{T}_H.
\end{aligned}$$

- Assumption: the number of element in w_K is less than N^n.

Residual Functions

For $u(x) \in V_h$, let $u^{(0)}(x) := u(x)$.
For each colour c_k, a residual function $u^{(k)}(x)$ is defined as follows

$$\begin{aligned} w^{(k)} &= I^H u^{(k-1)}, && (w^{(k)} \in V_H) \\ v^{(k)} &= u^{(k-1)} - w^{(k)}, && (v^{(k)} \in V_h) \\ v_i^{(k)} &= I_i^{h,H} \theta_i^{(c_k)} v^{(k)}, && (v_i^{(k)} \in V_i). \\ u^{(k)} &= v^{(k)} - \sum_{i \in \mathcal{C}_k} v_i^{(k)}, && (u^{(k)} \in V_h) \end{aligned}$$

Then the following equalities hold

$$u^{(k)}|_{\bar{\Omega}_i} \equiv 0, \quad \text{for all } i \in \mathcal{C}_{k_i},\ k_i \leq k,$$

$$u = \sum_{k=0}^{N^c-1} w^{(k)} + \sum_{k=1}^{N^c} \sum_{i \in C_k} v_i^{(k)},$$

$$\left| \sum_{i \in \mathcal{C}_k} v_i^{(k)} \right|_{H^1(\Omega)}^2 = \sum_{i \in \mathcal{C}_k} \left| v_i^{(k)} \right|_{H^1(\Omega)}^2 .$$

Square Problem

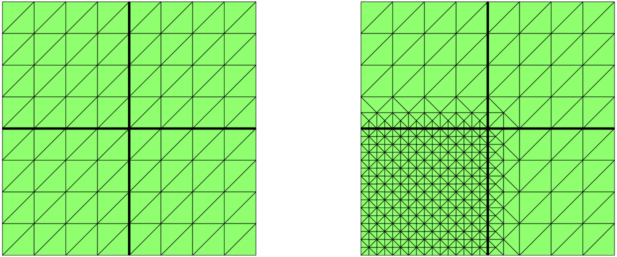
$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega = (0,1)\times(0,1),\\
u &= 0 && \text{on } \partial_D \Omega,
\end{aligned}
$$

We compute the smallest and second largest eigenvalues of $P^{-1}A$ for

$$H = 2^{-k},\ k = 2, 3, \ldots 6,$$

$$N = 2^{2l},\ 1 \le l \le k,$$

$$h = 2^{-m},\ m > k.$$



h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$N=4$	$H=2^{-2}$					
$\hat{\lambda}_{\min}$	0.9428	0.9303	0.9287	0.9283	0.9282	0.9282
$\hat{\lambda}_{\max}$	3.1389	4.0000	4.0000	4.0000	4.0000	4.0000
$N=16$						
$\hat{\lambda}_{\min}$	1.0000	0.9354	0.9290	0.9285	0.9283	0.9282
$\hat{\lambda}_{\max}$	5.4898	9.3334	9.3352	9.3353	9.3353	9.3353
$N=4$	$H=2^{-3}$					
$\hat{\lambda}_{\min}$		0.9304	0.9287	0.9283	0.9282	0.9282
$\hat{\lambda}_{\max}$		3.1360	4.0000	4.0000	4.0000	4.0000
$N=16$						
$\hat{\lambda}_{\min}$		0.9355	0.9290	0.9285	0.9283	0.9282
$\hat{\lambda}_{\max}$		3.2546	4.4092	4.4203	4.4207	4.4207
$N=64$						
$\hat{\lambda}_{\min}$		1.0000	0.9330	0.9286	0.9284	0.9283
$\hat{\lambda}_{\max}$		5.6720	9.9448	9.9509	9.9511	9.9507

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$N=4$			$H=2^{-4}$			
$\hat{\lambda}_{\min}$			0.9287	0.9283	0.9282	0.9282
$\hat{\lambda}_{\max}$			3.1355	4.0000	4.0000	4.0000
$N=16$						
$\hat{\lambda}_{\min}$			0.9291	0.9285	0.9283	0.9282
$\hat{\lambda}_{\max}$			3.1507	4.0229	4.0242	4.0243
$N=64$						
$\hat{\lambda}_{\min}$			0.9331	0.9286	0.9284	0.9282
$\hat{\lambda}_{\max}$			3.3108	4.4541	4.4657	4.4661
$N=256$						
$\hat{\lambda}_{\min}$			1.0000	0.9324	0.9285	0.9284
$\hat{\lambda}_{\max}$			5.7366	10.1181	10.1248	10.1251

h	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}
$N = 4$				$H = 2^{-5}$		
$\hat{\lambda}_{\min}$				0.9284	0.9282	0.9282
$\hat{\lambda}_{\max}$				3.1355	4.0000	4.0000
$N = 16$						
$\hat{\lambda}_{\min}$				0.9285	0.9283	0.9282
$\hat{\lambda}_{\max}$				3.1366	4.0012	4.0013
$N = 64$						
$\hat{\lambda}_{\min}$				0.9286	0.9284	0.9283
$\hat{\lambda}_{\max}$				3.1541	4.0253	4.0270
$N = 256$						
$\hat{\lambda}_{\min}$				0.9325	0.9285	0.9284
$\hat{\lambda}_{\max}$				3.3270	4.4638	4.4754
$N = 1024$						
$\hat{\lambda}_{\min}$				1.0000	0.9322	0.9285
$\hat{\lambda}_{\max}$				5.7671	10.1580	10.1647

Coloring

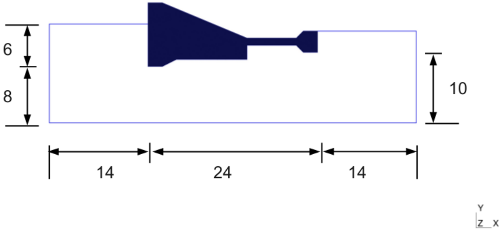
1	2	4	3
4	3	1	2
1	2	4	3
4	3	1	2

1	2	5	1
4	3	6	4
7	8	9	7
1	2	5	1

Seepage Under Dam

$$
\begin{aligned}
-\nabla \cdot (k(x,y)\nabla h(x,y)) &= 0 && \text{in } \Omega, \\
h &= 10 && \text{on } \partial_{D1}\Omega, \\
h &= 1 && \text{on } \partial_{D2}\Omega, \\
\frac{\partial h}{\partial n} &= 0 && \text{on } \partial_{N}\Omega,
\end{aligned}
$$

where $h(x, y)$ is the total head and $k(x, y)$ is the hydraulic permeability coefficient



(courtesy of Dr. Waad Subber)

Seepage Under Dam: Coarse Mesh of 1264 DOFs

	$n = 5832$		$n = 22931$		$n = 90933$		$n = 362153$		$n = 1445457$	
N	CG	GM	CG	GM	CG	GM	CG	GM	CG	GM
2^1	3(9)	2(7)	3(9)	2(8)	3(8)	2(7)	3(9)	2(7)	3(9)	3(8)
2^2	4(10)	3(9)	4(10)	3(9)	4(9)	3(8)	4(10)	3(9)	4(11)	3(9)
2^3	6(11)	4(10)	6(11)	4(10)	6(10)	4(9)	5(11)	4(10)	5(13)	4(11)
2^4	8(12)	5(11)	8(11)	5(10)	8(11)	5(10)	8(13)	5(11)	7(16)	5(13)
2^5	10(12)	4(12)	10(12)	5(10)	10(12)	5(11)	10(14)	5(13)	9(18)	5(16)
2^6	11(14)	5(13)	12(12)	5(11)	11(13)	5(11)	11(15)	5(14)	10(19)	6(16)
2^7	12(15)	5(14)	13(13)	5(12)	13(14)	6(12)	12(17)	6(15)	12(21)	6(18)
2^8	16(16)	7(15)	16(15)	7(13)	16(16)	7(14)	16(19)	7(16)	15(23)	7(20)
2^9	16(17)	8(16)	18(15)	8(13)	18(16)	8(15)	18(20)	8(17)	17(26)	8(21)

Number of CG iterations and GMRES iterations to reduce the error and residual respectively by a factor of $1e6$ using P^{-1} and two-level additive Schwarz (in parentheses).

Seepage Under Dam: Coarse Mesh of 2784 DOFs

	$n = 10859$		$n = 42885$		$n = 170441$		$n = 679569$		$n = 2713889$	
N	CG	GM	CG	GM	CG	GM	CG	GM	CG	GM
2^1	3(9)	2(7)	3(9)	2(8)	3(8)	2(7)	3(8)	2(7)	3(9)	2(8)
2^2	4(10)	3(9)	4(9)	3(8)	4(9)	3(8)	4(10)	3(8)	4(11)	3(9)
2^3	6(11)	3(10)	6(10)	3(9)	6(11)	3(9)	6(12)	3(11)	6(15)	3(13)
2^4	8(12)	4(11)	8(11)	5(10)	8(12)	5(10)	8(14)	5(11)	7(17)	5(14)
2^5	9(12)	4(11)	9(11)	5(10)	9(12)	5(10)	9(14)	5(12)	9(17)	5(15)
2^6	10(13)	4(12)	10(12)	4(11)	10(13)	5(11)	10(16)	5(14)	10(20)	6(16)
2^7	11(14)	4(13)	12(13)	4(11)	11(14)	4(11)	11(16)	4(14)	10(20)	4(17)
2^8	14(15)	6(14)	15(13)	6(12)	15(15)	6(13)	15(18)	6(15)	14(23)	6(20)
2^9	17(17)	7(16)	17(15)	7(13)	17(16)	7(13)	17(19)	7(16)	16(24)	7(20)

Number of CG iterations and GMRES iterations to reduce the error and residual respectively by a factor of $1e6$ using A^{-1} and two-level additive Schwarz (in parentheses).

Conclusions

- We formulated a preconditioner for parallel adaptive finite elements
- We showed that the preconditioner is optimal (the effective condition number is bounded independently of the mesh sizes and the number of subdomains.
- Numerical experiments confirm the theoretical results.

Acknowledgement

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Scottish Funding Council
Promoting further and higher education

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