Schwarz Preconditioner for the Stochastic Finite Element Method

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Introduction

- Motivation
  - High resolution numerical model
    - effectively reduces discretization error.
    - does not necessarily enhance confidence in prediction.
  - The effect of uncertainty need to be considered for realistic computer predictions.
- Objective
  - Develop parallel algorithms to quantify uncertainty in large-scale computational models.
- Methodology
  - Exploit domain decomposition methods in the spatial direction in conjunction with a functional expansion along the stochastic dimension.

Uncertainty Propagation and Data Assimilation

- Model Equation

$$\mathbf{u}_{k+1} = \boldsymbol{\psi}_k\left(\mathbf{u}_k, \mathbf{f}_k, \mathbf{q}_k\right) \quad -- \text{ Forecast Step}$$

- Measurement Equation

$$\mathbf{d}_k = \mathbf{h}_k\left(\mathbf{u}_k, \boldsymbol{\epsilon}_k\right) \qquad\qquad -- \text{ Assimilation Step}$$

Uncertainty Propagation

- Traditional Monte Carlo Simulation
  - Non-intrusive to legacy code
  - Embarrassingly parallel yet computationally expensive
- Polynomial Chaos Expansion
  - Intrusive or non-intrusive
  - Computationally efficient
  - Multiscale representation of uncertainty

Stochastic Elliptic PDE

- Find a random function $u(\mathbf{x},\theta) : D \times \Omega \to \mathbb{R}$ satisfying the following equation in an almost surely sense:

$$\begin{aligned} \nabla \cdot (\kappa(\mathbf{x},\theta)\nabla u(\mathbf{x},\theta)) &= f(\mathbf{x}), \quad \text{in} \quad D \times \Omega, \\ u(\mathbf{x},\theta) &= 0, \quad \text{on} \quad \partial D \times \Omega, \end{aligned}$$

$$0 < \kappa_{min} \leq \kappa(\mathbf{x},\theta) \leq \kappa_{max} < +\infty, \quad \text{in} \quad D \times \Omega.$$

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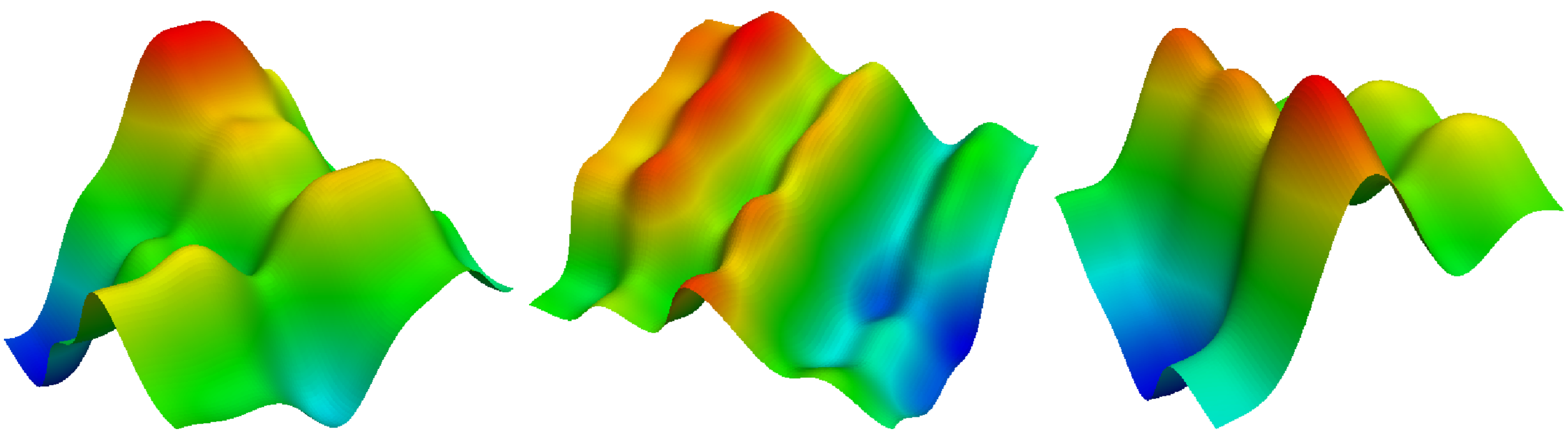
u(\mathbf{x},\theta) &= 0, \quad \text{on} \quad \partial D \times \Omega,
\end{aligned}
$$

$$
0 < \kappa_{min} \le \kappa(\mathbf{x},\theta) \le \kappa_{max} < +\infty, \quad \text{in} \quad D \times \Omega.
$$

- Possible realizations of $\kappa(\mathbf{x},\theta)$:



Uncertainty Representation by Stochastic Processes

- Karhunen-Loeve Expansion (KLE)

$$\kappa(\mathbf{x}, \theta) = \bar{\kappa}(\mathbf{x}) + \sum_{i=1}^{M} \xi_i(\theta)\sqrt{\lambda_i}\phi_i(\mathbf{x}),$$

$$\langle \xi_i(\theta) \rangle = 0, \quad \langle \xi_i(\theta)\xi_j(\theta) \rangle = \delta_{ij}.$$

- Fredholm Integral Equation

$$\int C_{\kappa\kappa}(\mathbf{x}_1, \mathbf{x}_2)\phi_i(\mathbf{x}_1)d\mathbf{x}_1 = \lambda_i\phi_i(\mathbf{x}_2).$$

Uncertainty Representation by Stochastic Processes

- Polynomial Chaos Expansion (PCE)

$$u(\mathbf{x},\theta) = \sum_{i=0}^{N} u_i(\mathbf{x})\Psi_i(\boldsymbol{\xi}),$$

$$\langle \Psi_i(\boldsymbol{\xi}) \rangle = 0, \quad \langle \Psi_i(\boldsymbol{\xi})\Psi_j(\boldsymbol{\xi}) \rangle = \delta_{ij} \langle \Psi_i^2(\boldsymbol{\xi}) \rangle, \quad N+1 = \frac{(M+p)!}{M!p!}.$$

- Two-dimensional $(M=2)$ third order $(p=3)$ PCE

$$\begin{aligned} u(\mathbf{x},\theta) = {} & u_0(\mathbf{x}) + \\ & u_1(\mathbf{x})\xi_1 + u_2(\mathbf{x})\xi_2 + \\ & u_3(\mathbf{x})(\xi_1^2 - 1) + u_4(\mathbf{x})(\xi_1\xi_2) + u_5(\mathbf{x})(\xi_2^2 - 1) + \\ & u_6(\mathbf{x})(\xi_1^3 - 3\xi_1) + u_7(\mathbf{x})(\xi_1^2\xi_2 - \xi_2) + u_8(\mathbf{x})(\xi_1\xi_2^2 - \xi_1) + u_9(\mathbf{x})(\xi_2^3 - 3\xi_2). \end{aligned}$$

The Spectral Stochastic FEM

- The FEM discretization of an SPDE

$$\mathbf{A}(\kappa(\theta))\mathbf{u}(\theta) = \mathbf{f}.$$

- Expanding system parameters and solution process by KLE and PCE

$$\kappa(\theta) = \sum_{i=0}^{M} \xi_i \kappa_i, \quad \text{and} \quad \mathbf{u}(\theta) = \sum_{j=0}^{N} \Psi_j \mathbf{u}_j$$

- Galerkin projection

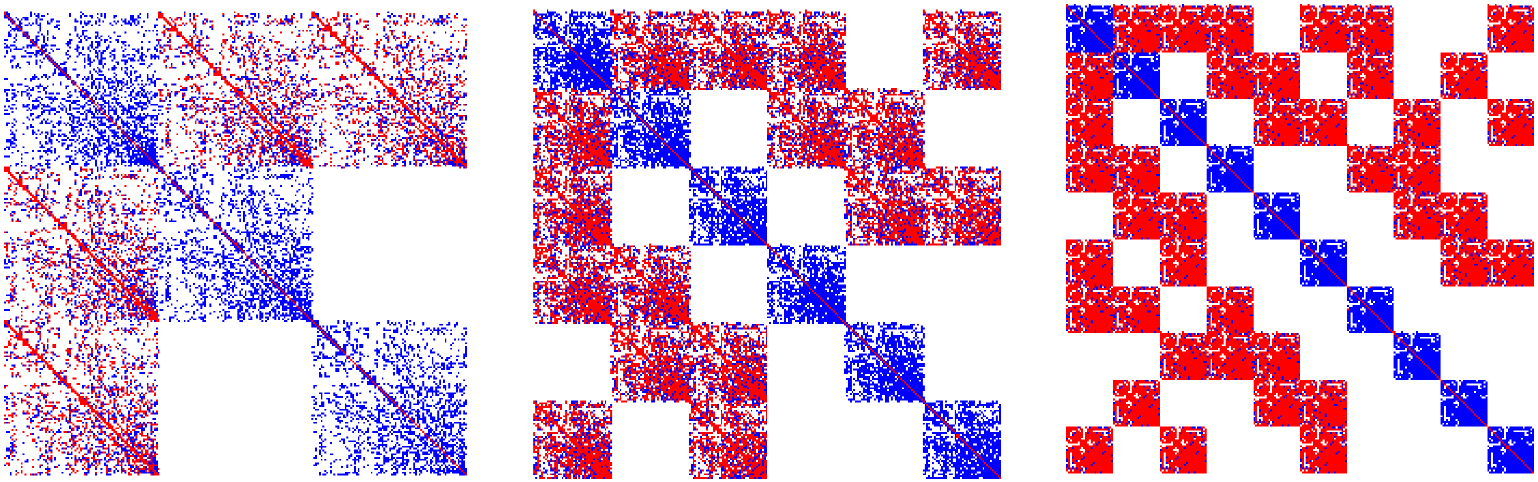
$$\mathcal{A}\mathcal{U} = \mathcal{F},$$

where

$$\mathcal{A} = \sum_{i=0}^{M} \mathbf{C}_i \otimes \mathbf{A}_i, \quad \mathbf{C}_{ijk} = \langle \xi_i \Psi_j \Psi_k \rangle \quad \text{and} \quad \mathcal{F}_k = \langle \Psi_k \mathbf{f} \rangle.$$

The Spectral Stochastic FEM

$$\mathcal{A}\mathcal{U} = \mathcal{F}$$



(a) 1st (b) 2nd (c) 3rd

- Large-scale linear system
- Very ill-conditioned
- Simple iterative methods are inefficient!
- Sparse and block structured
- Symmetric positive-definite (for elliptic SPDEs)

Preconditioned Conjugate Gradient Method (PCGM)

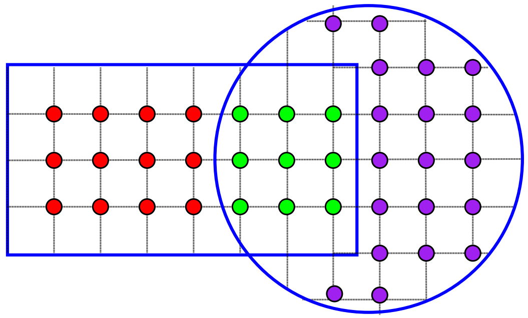
$$\mathcal{A}\,\mathcal{U} = \mathcal{F},$$

$$\mathcal{M}^{-1}\mathcal{A}\,\mathcal{U} = \mathcal{M}^{-1}\mathcal{F},$$

- $(\mathcal{M}^{-1})$ is a good approximation to $(\mathcal{A}^{-1})$
- Condition number of $(\mathcal{M}^{-1}\mathcal{A})$ is much smaller than $(\mathcal{A})$
- Eigenvalues of $(\mathcal{M}^{-1}\mathcal{A})$ are clustered near one

Schwarz preconditioner for stochastic PDEs

- Partition the spatial domain



- Define a restiriction matrix

$$\mathbf{R}_s^T : \Omega_s \mapsto \Omega$$

- For each of KLE coefficient, define the subdomain stiffness matrix

$$\mathbf{A}_i^s = \mathbf{R}_s \mathbf{A}_i \mathbf{R}_s^T$$

- Corresponds to

$$\begin{aligned} \nabla \cdot (\kappa_i(\mathbf{x}) \nabla u(\mathbf{x})) &= f(\mathbf{x}), \quad \text{in} \quad D_s, \\ u(\mathbf{x}) &= g(\mathbf{x}), \quad \text{on} \quad \partial D_s. \end{aligned}$$

Schwarz preconditioner for stochastic PDEs

- The subdomain stochastic stiffness matrix

$$\mathcal{A}_s = \sum_{i=0}^{M} \mathbf{C}^i \otimes \mathbf{A}_i^s,$$

- Inspired by the Schwarz theory, we define the one-level stochastic Schwarz preconditioner

$$\mathcal{M}^{-1} = \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s.$$

Schwarz preconditioner for stochastic PDEs

- The mean-based stochastic Schwarz preconditioner

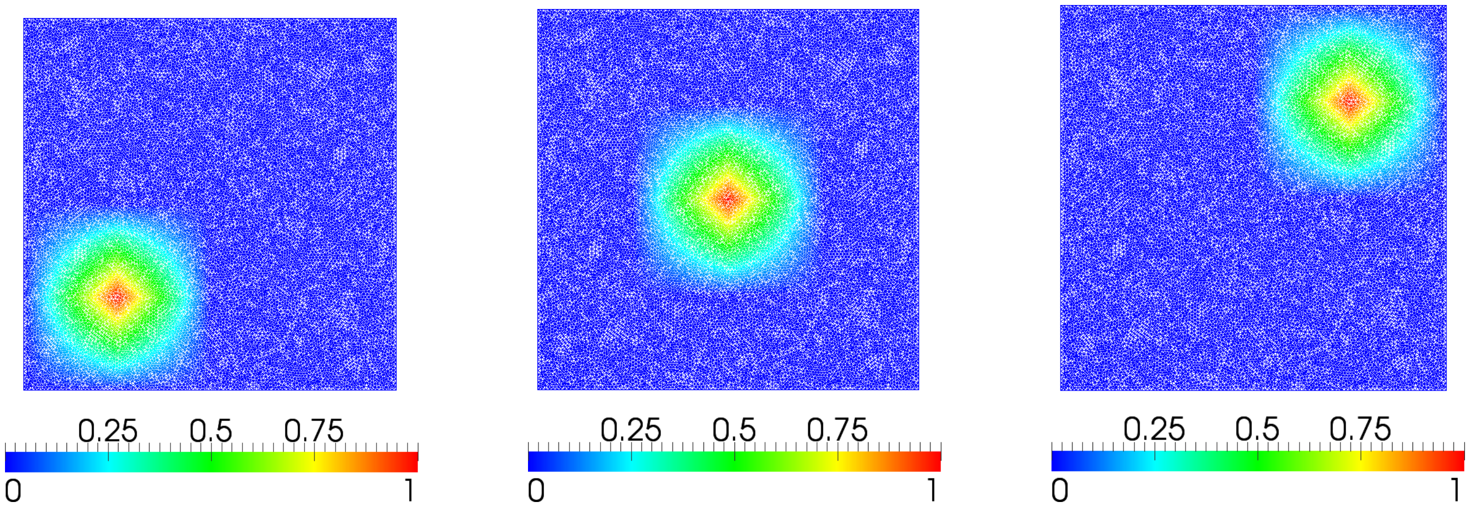
$$\mathcal{M}_0^{-1} = [\mathbf{C}^0]^{-1} \otimes \sum_{s=1}^{S} \mathbf{R}_s^T [\mathbf{A}_s^0]^{-1} \mathbf{R}_s,$$

- Generalization of the block-diagonal mean-based preconditioner (Powell IMAJNA 2009, Pellissetti AES 2000, Ghanem CMAME 1996, others..)

$$\mathcal{M}_0^{-1} = \mathbf{I} \otimes [\mathbf{A}^0]^{-1}.$$

Coarse Grid Correction

- Define a set of bilinear hat basis functions



- A coarse grid restiriction operator is defined as

$$\mathbf{R}_0^T = \begin{bmatrix} \psi_1(\mathbf{x}_1) & \psi_2(\mathbf{x}_1) & \cdots & \psi_{n_0}(\mathbf{x}_1) \\ \psi_1(\mathbf{x}_2) & \psi_2(\mathbf{x}_2) & \cdots & \psi_{n_0}(\mathbf{x}_2) \\ \vdots & \vdots & \cdots & \vdots \\ \psi_1(\mathbf{x}_{n_i}) & \psi_2(\mathbf{x}_{n_i}) & \cdots & \psi_{n_0}(\mathbf{x}_{n_i}) \end{bmatrix}$$

Stochastic additive Schwarz preconditioner

- Two-level Schwarz preconditioner for SPDEs

$$\mathcal{M}^{-1} = \mathcal{R}_0^T \mathcal{A}_0^{-1} \mathcal{R}_0 + \sum_{s=1}^{S} \mathcal{R}_s^T \mathcal{A}_s^{-1} \mathcal{R}_s,$$

- **Theorem**: The condition number of the stochastic additive Schwarz preconditioner is bounded by

$$cond\left(\mathcal{M}^{-1}\mathcal{A}\right) \leq C \frac{\kappa_{max}}{\kappa_{min}} \left(1 + \frac{H}{h}\right).$$

where C is a constant independent of H, h and δ, M,and p.

Numerical Results

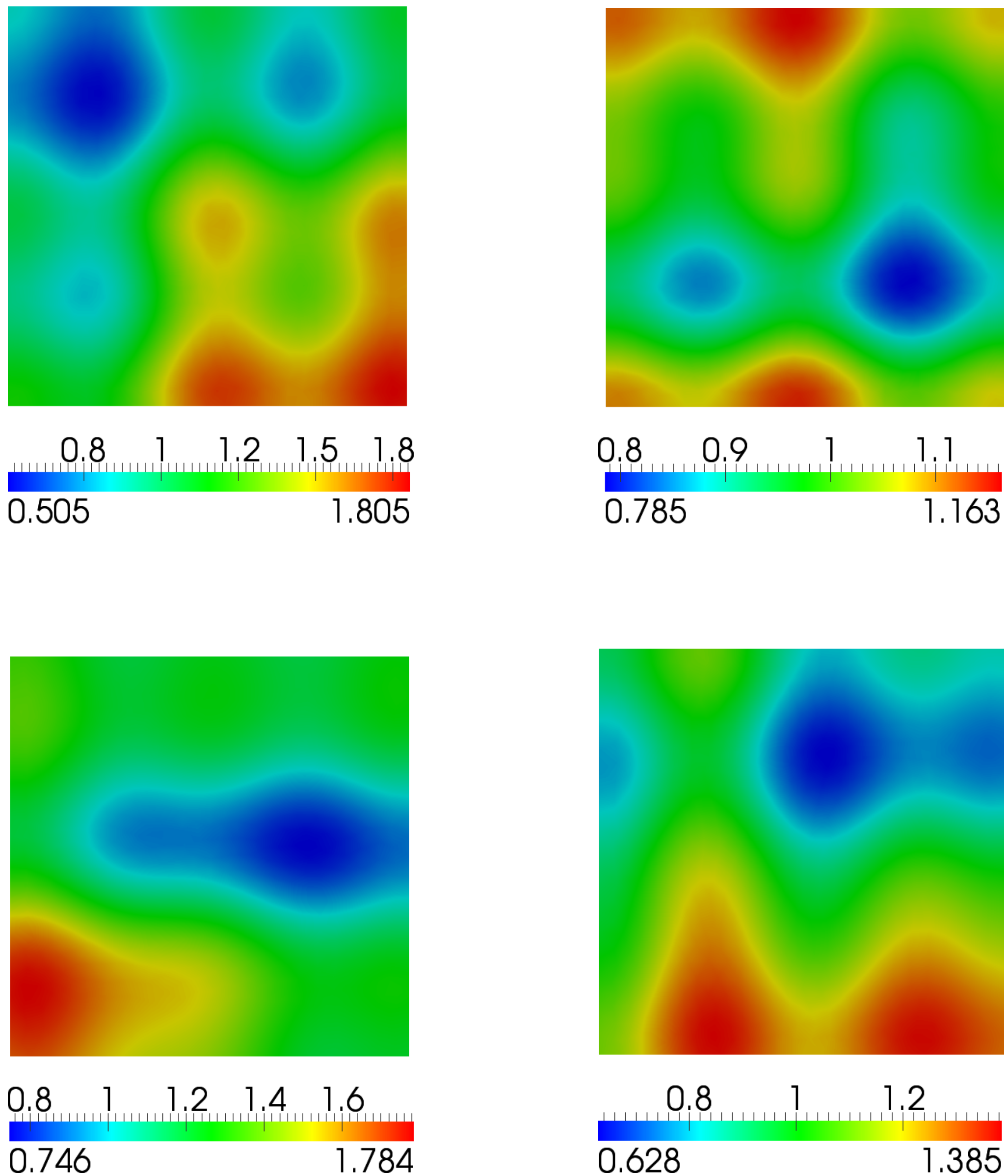
- Poisson's equation with random coefficient

$$\nabla \cdot (\kappa(\mathbf{x},\theta)\nabla u(\mathbf{x},\theta)) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$
$$u(\mathbf{x},\theta) = 0, \quad \mathbf{x} \in \partial\Omega.$$

  - The permeability coefficient is a Gaussian or Uniform stochastic process with an exponential covariance

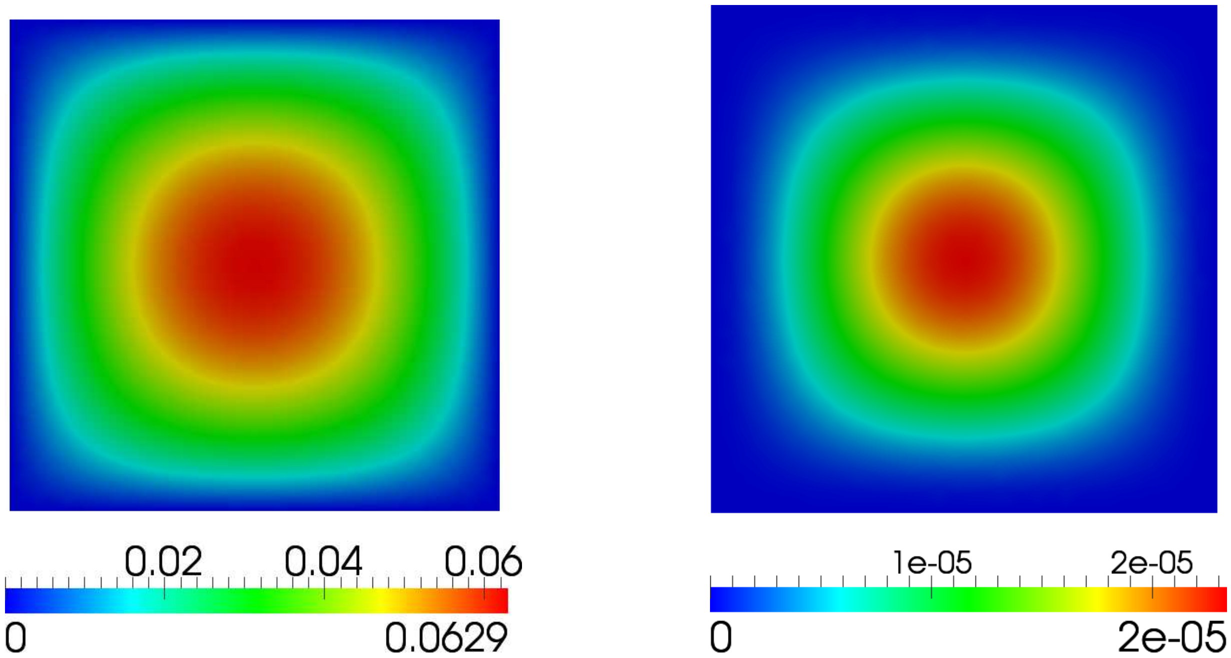
$$C_{\kappa\kappa}(\mathbf{x},\mathbf{y}) = \sigma^2 \exp\left(\frac{-|x_1 - y_1|}{b_1} + \frac{-|x_2 - y_2|}{b_2}\right).$$

Realizations of the permeability coefficient $\kappa(\mathbf{x}, \theta)$



Stochastic Features

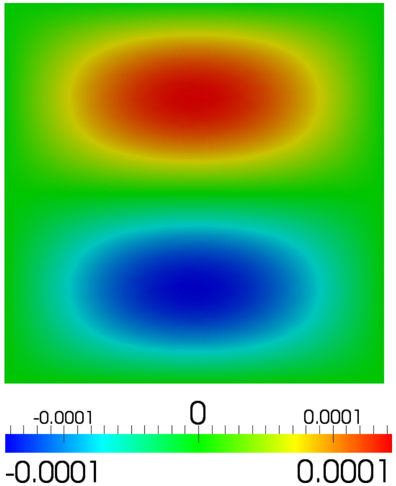
- Mean and Variance of the solution process



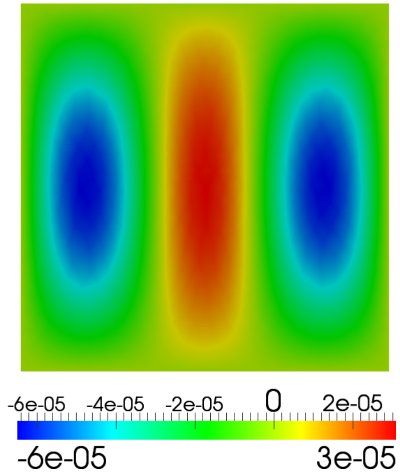
(a) μ_u (b) σ_u^2

Stochastic Features

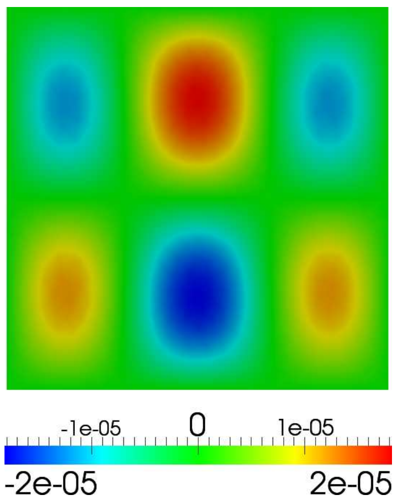
- Selected PCE coefficients of the solution process



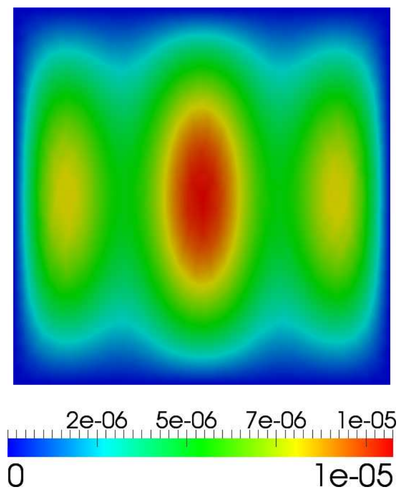
(a) $\mathbf{u}_7$



(b) $\mathbf{u}_8$



(a) $\mathbf{u}_{13}$



(b) $\mathbf{u}_{14}$

Numerical Results

- Scalability with respect to dimension and order of the PCE

		Gaussian		Uniform	
M	p	$cond$	$iter$	$cond$	$iter$
2	1	9.7092	21	9.7027	21
	2	9.7195	21	9.7060	21
	3	9.7272	21	9.7077	21
	4	9.7335	21	9.7087	21

Numerical Results

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M	p	$cond$	$iter$	$cond$	$iter$
2	1	9.7092	21	9.7027	21
	2	9.7195	21	9.7060	21
	3	9.7272	21	9.7077	21
	4	9.7335	21	9.7087	21
3	1	9.7097	21	9.7029	21
	2	9.7203	21	9.7068	21
	3	9.7283	21	9.7090	21
	4	9.7347	21	9.7104	21

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	4	9.7335	21	9.7087	21
3	1	9.7097	21	9.7029	21
	2	9.7203	21	9.7068	21
	3	9.7283	21	9.7090	21
	4	9.7347	21	9.7104	21
4	1	9.7408	21	9.7207	21
	2	9.7751	21	9.7326	21
	3	9.8031	21	9.7391	21
	4	9.8273	21	9.7431	21

Numerical Results

- Scalability with respect to the strength of randomness

		Gaussian		Uniform	
$\frac{\sigma}{\mu}$	p	$cond$	$iter$	$cond$	$iter$
0.1	1	9.7408	21	9.7207	21
	2	9.7751	21	9.7326	21
	3	9.8031	21	9.7391	21
	4	9.8273	21	9.7431	21

Numerical Results

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	2	9.7751	21	9.7326	21
	3	9.8031	21	9.7391	21
	4	9.8273	21	9.7431	21
0.2	1	9.7885	21	9.7482	21
	2	9.8568	22	9.7704	21
	3	9.9134	22	9.7820	21
	4	9.9638	22	9.7891	21

Numerical Results

- Scalability with respect to the strength of randomness

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$\frac{\sigma}{\mu}$	p	$cond$	$iter$	$cond$	$iter$
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	3	9.8031	21	9.7391	21
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0.2	1	9.7885	21	9.7482	21
	2	9.8568	22	9.7704	21
	3	9.9134	22	9.7820	21
	4	9.9638	22	9.7891	21
0.3	1	9.8367	22	9.7757	21
	2	9.9414	22	9.8071	21
	3	10.0338	22	9.8227	21
	4	10.1258	22	9.8321	21

Numerical Results

- Scalability with respect to the overlap

		Gaussian		Uniform	
δ	p	$cond$	$iter$	$cond$	$iter$
$2h$	1	7.3252	16	7.3141	16
	2	7.3441	16	7.3203	16
	3	7.3593	16	7.3236	16
	4	7.3724	16	7.3248	16

Numerical Results

- Scalability with respect to the overlap

		Gaussian		Uniform	
δ	p	$cond$	$iter$	$cond$	$iter$
$2h$	1	7.3252	16	7.3141	16
	2	7.3441	16	7.3203	16
	3	7.3593	16	7.3236	16
	4	7.3724	16	7.3248	16
$3h$	1	5.2401	13	5.2371	13
	2	5.2448	14	5.2391	13
	3	5.2485	14	5.2401	13
	4	5.2517	14	5.2522	13

Numerical Results

- Scalability with respect to the overlap

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δ	p	$cond$	$iter$	$cond$	$iter$
$2h$	1	7.3252	16	7.3141	16
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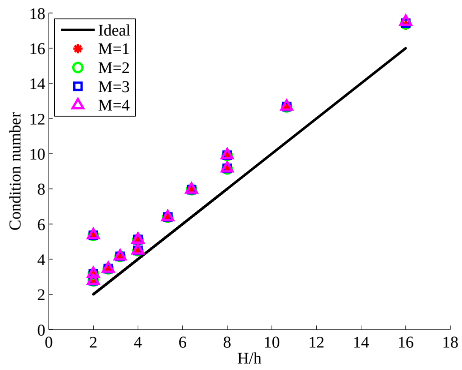
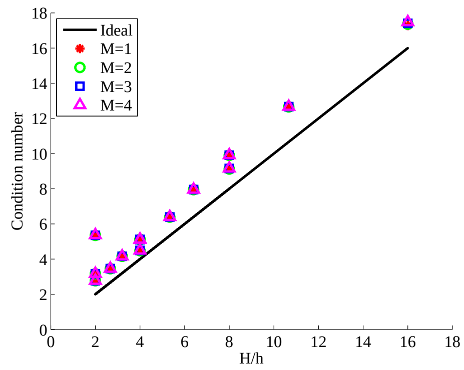
	3	7.3593	16	7.3236	16
	4	7.3724	16	7.3248	16
$3h$	1	5.2401	13	5.2371	13
	2	5.2448	14	5.2391	13
	3	5.2485	14	5.2401	13
	4	5.2517	14	5.2522	13
$4h$	1	4.6796	12	4.6767	12
	2	4.6843	13	4.6785	13
	3	4.6881	13	4.6794	13
	4	4.6913	13	4.6801	13

Numerical Results

- Scalability with respect to $\frac{H}{h}$: fixed overlap
  - $p = 2$

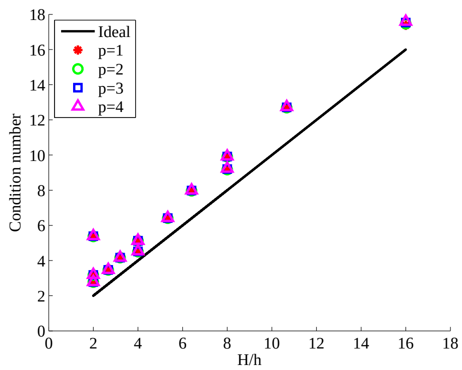


(a) Gaussian

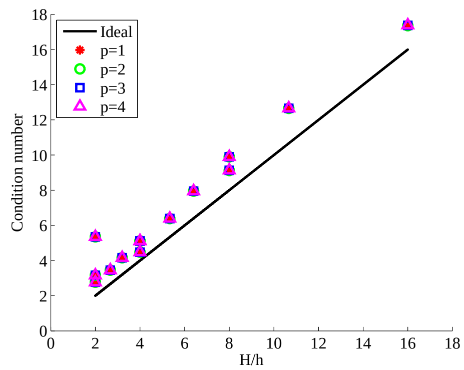


(b) Uniform

  - $M = 2$



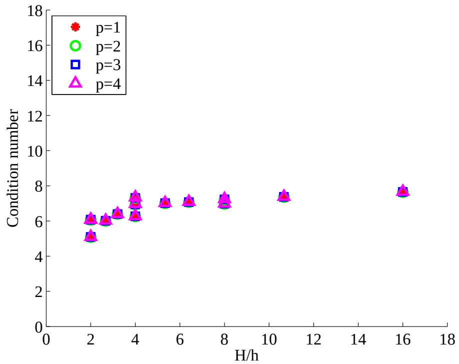
(c) Gaussian



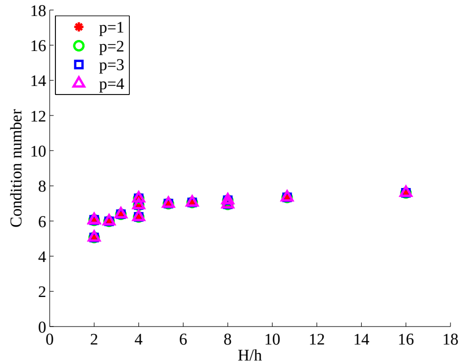
(d) Uniform

Numerical Results

- Scalability with respect to $\frac{H}{h}$: proportional overlap $\delta = 0.1H$
  - $p = 2$

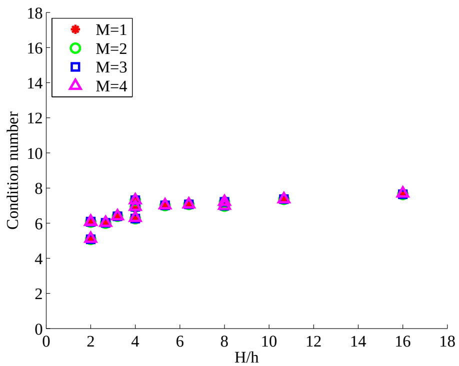


(a) Gaussian

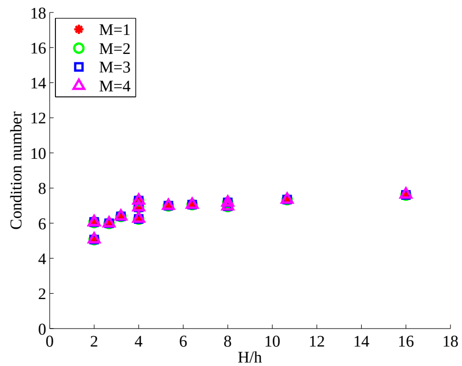


(b) Uniform

  - $M = 2$



(c) Gaussian



(d) Uniform

Numerical Results

- Two-Dimensional Elasticity Problem

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma}\left(\mathbf{u}(\mathbf{x},\theta)\right) &= \mathbf{f}(\mathbf{x}), && \text{in} && \Omega \times \Theta, \\ \boldsymbol{\sigma}\left(\mathbf{u}(\mathbf{x},\theta)\right) \cdot \mathbf{n} &= \mathbf{g}(\mathbf{x}), && \text{on} && \partial\Omega_N \times \Theta, \\ \mathbf{u}(\mathbf{x},\theta) &= 0, && \text{on} && \partial\Omega_D \times \Theta. \end{aligned}$$

  - Hooke's law

$$\boldsymbol{\sigma}\left(\mathbf{u}(\mathbf{x},\theta)\right) = 2\mu\, \boldsymbol{\epsilon}\left(\mathbf{u}(\mathbf{x},\theta)\right) + \lambda\, \mathrm{tr}\left(\boldsymbol{\epsilon}\left(\mathbf{u}(\mathbf{x},\theta)\right)\right) \mathbf{I}.$$

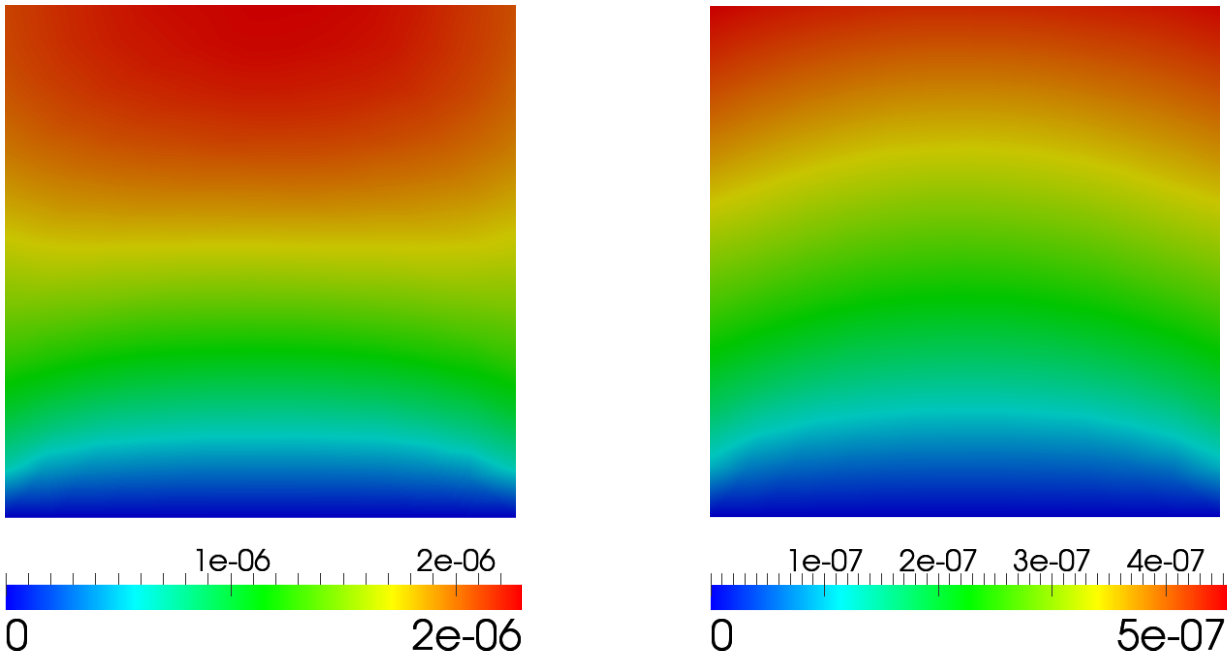
  - Lam$\acute{e}$ constants

$$\lambda = \frac{E(\mathbf{x},\theta)}{2(1+\nu)}, \qquad \mu = \frac{E(\mathbf{x},\theta)\nu}{(1+\nu)(1-2\nu)}.$$

  - Young's modulus $E(\mathbf{x},\theta)$ is modeled as a random field and Poisson's ratio ν is assumed to be deterministic quantity

Stochastic Features

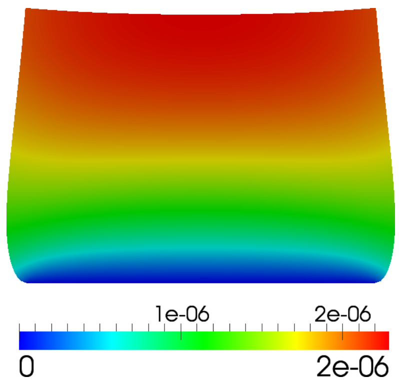
- Mean and standard deviation of the displacement field



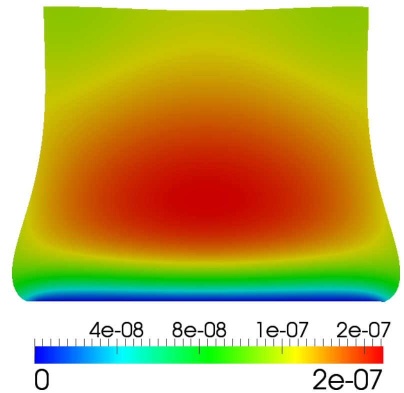
(a) μ_u (b) σ_u

Stochastic Features

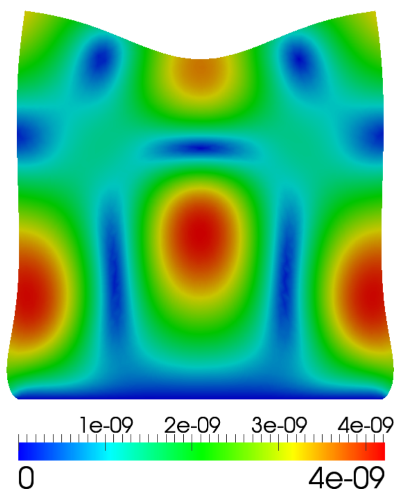
- Deformed meshes corresponding to chaos coefficients



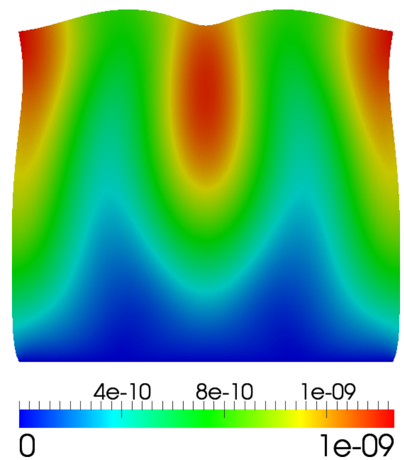
(a) $\mathbf{u}_0$



(b) $\mathbf{u}_3$



(a) $\mathbf{u}_9$



(b) $\mathbf{u}_{14}$

Numerical Results

- Scalability with respect to dimension and order of the PCE

		Gaussian		Uniform	
M	p	$cond$	$iter$	$cond$	$iter$
2	1	13.6048	23	13.5442	22
	2	13.7231	24	13.5699	22
	3	13.8751	25	13.5883	22
	4	14.0697	25	13.6019	22

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	3	13.8751	25	13.5883	22
	4	14.0697	25	13.6019	22
3	1	13.6563	22	13.5760	22
	2	13.8143	24	13.6282	22
	3	14.0014	25	13.6614	23
	4	14.2790	25	13.6851	23

Numerical Results

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	2	13.7231	24	13.5699	22
	3	13.8751	25	13.5883	22
	4	14.0697	25	13.6019	22
3	1	13.6563	22	13.5760	22
	2	13.8143	24	13.6282	22
	3	14.0014	25	13.6614	23
	4	14.2790	25	13.6851	23
4	1	13.6953	24	13.5981	22
	2	13.8824	25	13.6681	23
	3	14.0936	25	13.7173	23
	4	14.3575	25	13.7525	23

Numerical Results

- Scalability with respect to the strength of randomness

		Gaussian		Uniform	
$\frac{\sigma}{\mu}$	p	$cond$	$iter$	$cond$	$iter$
0.1	1	13.5623	22	13.5307	22
	2	13.6160	22	13.5546	22
	3	13.6626	24	13.5702	22
	4	13.7056	24	13.5806	22

Numerical Results

- Scalability with respect to the strength of randomness

		Gaussian		Uniform	
$\frac{\sigma}{\mu}$	p	$cond$	$iter$	$cond$	$iter$
0.1	1	13.5623	22	13.5307	22
	2	13.6160	22	13.5546	22
	3	13.6626	24	13.5702	22
	4	13.7056	24	13.5806	22
0.2	1	13.6472	23	13.5745	22
	2	13.7786	24	13.6274	22
	3	13.9120	24	13.6636	22
	4	14.0581	24	13.6888	22

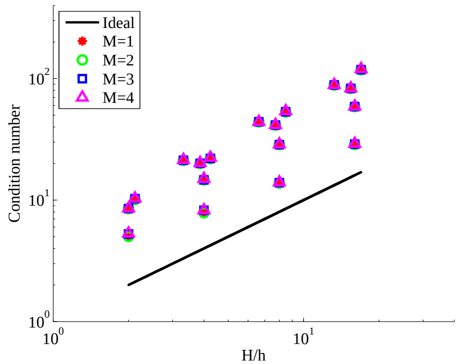
Numerical Results

- Scalability with respect to the strength of randomness

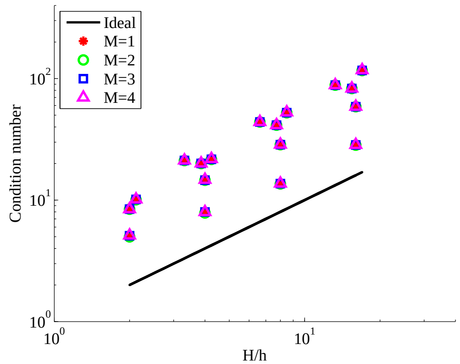
		Gaussian		Uniform	
$\frac{\sigma}{\mu}$	p	$cond$	$iter$	$cond$	$iter$
0.1	1	13.5623	22	13.5307	22
	2	13.6160	22	13.5546	22
	3	13.6626	24	13.5702	22
	4	13.7056	24	13.5806	22
0.2	1	13.6472	23	13.5745	22
	2	13.7786	24	13.6274	22
	3	13.9120	24	13.6636	22
	4	14.0581	24	13.6888	22
0.3	1	13.7478	24	13.6230	23
	2	14.0075	25	13.7122	23
	3	14.3430	25	13.7767	23
	4	15.0671	28	13.8243	23

Numerical Results

- Scalability with respect to $\frac{H}{h}$
  - $p = 2$

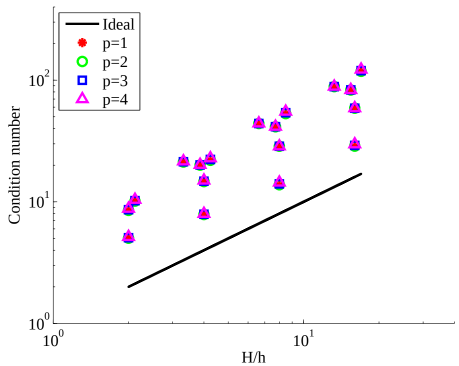


(a) Gaussian

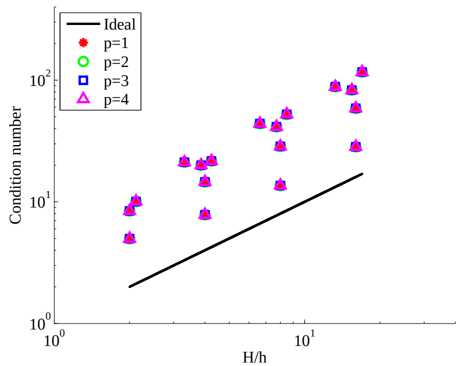


(b) Uniform

  - $M = 2$



(c) Gaussian



(d) Uniform

Conclusions

- Two-level Schwarz domain decomposition preconditioner is introduced for the linear system of the spectral stochastic finite element method.
  - The stochastic Schwarz preconditioner achieves a convergence rate that is independent of the coefficient of variation, dimension and order of the stochastic expansion.
  - The condition number of the stochastic Schwarz preconditioner grows as $\mathcal{O}\left(\frac{H}{h}\right)$.

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