From core-collapse supernova neutrino transport to fuzzy domain decomposition methods

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joint work with
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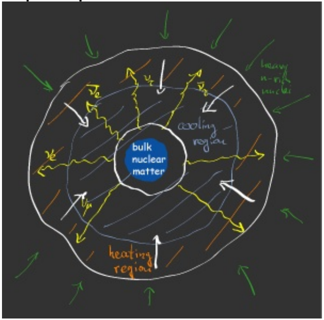
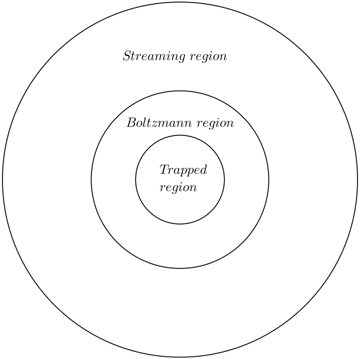
Motivation

Complex problems with multiscale physics



thanks to Liebendörfer



- Too costly to solve exactly ! $\Rightarrow$ Need approximations
- Different asymptotic in different regions
- $\Rightarrow$ Heterogeneous DD methods
- Validity domain of the asymptotic not a priori known !
- Need of new solutions and new algorithms !

Comoving Boltzmann equation in spherical symmetry

Radiative Transfer

Boltzmann $\mathcal{O}(v/c)$ equation for distribution function $f(t, r, \mu, \omega)$ in comoving frame

$$\frac{1}{c}\frac{df}{dt} + \mu\frac{\partial f}{\partial r} + F_\mu(\mathbf{u})\frac{\partial f}{\partial \mu} + F_\omega(\mathbf{u})\frac{\partial f}{\partial \omega} = j(\mathbf{u}) - \tilde{\chi}(\mathbf{u})f + \mathcal{C}(f, \mathbf{u})$$

with

$F_\mu = \left[\mu\left(\frac{d\ln\rho}{cdt} + \frac{3v}{cr}\right) + \frac{1}{r}\right](1-\mu^2)$ and $F_\omega = \left[\mu^2\left(\frac{d\ln\rho}{cdt} + \frac{3v}{cr}\right) - \frac{v}{cr}\right]\omega$.

- μ : cosine of the angle between radius and direction of neutrino propagation, r : radius, t : time, ω : neutrino energy,
- $\frac{d}{dt} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial r}$ Lagrangian time derivative
- c : speed of light, ρ : the fluid density, v : the fluid velocity,
- $j(r, t, \mu, \omega, \mathbf{u})$: emissivity, $\tilde{\chi}(r, t, \mu, \omega, \mathbf{u})$: absorptivity, $\mathbf{u}$: the fluid state vector
- $\mathcal{C}(f, \mathbf{u})$: isoenergetic scattering collision integral.

IDSA Equations

Notation: $\langle \cdot \rangle := \frac{1}{2}\int_{-1}^{1} \cdot \, d\mu$

IDSA equations

Under some appropriate assumptions, the dynamic of $\langle f \rangle$ can be approximated by $\langle f \rangle \approx \langle f^t \rangle + \langle f^s \rangle$ given by the following **IDSA** system

$$\frac{d\langle f^t \rangle}{cdt} + \frac{1}{3}\frac{d \ln \rho}{cdt}\omega\frac{\partial \langle f^t \rangle}{\partial \omega} = j - \tilde{\chi}\langle f^t \rangle - \Sigma,$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g \langle f^s \rangle\right) = -\tilde{\chi}\langle f^s \rangle + \Sigma,$$

$$\Sigma := \min\left\{\max\left[\frac{1}{r^2}\frac{\partial}{\partial r}\frac{-r^2\lambda}{3}\frac{\partial \langle f^t \rangle}{\partial r} + \tilde{\chi}\langle f^s \rangle,\ 0\right],\ j\right\},$$

where λ is the mean free path and $g(t, r, \omega)$ is a geometrical factor.

[Liebendörfer et al. 2009], [Berninger et al. 2012, 2013]

On the assumption $u = u_1 + u_2$

For a generic partial differential equation of the form

$$\mathcal{L}(u) = g \qquad \text{on } \Omega, \tag{1}$$

where $\mathcal{L}$ is a linear differential operator,

Assumption ($u = u_1 + u_2$)

We assume that the solution u of (1) can be written as a sum, $u = u_1 + u_2$, and that one can derive a coupled system for the new unknowns u_1 and u_2. The derivation of the coupled system might then use relevant approximations for one or both components.

- Neutrino radiative transfer in core-collapse supernovae.

  [Liebendörfer et al. 2009], [Berninger et al. 2012, 2013]

- Coupling of the kinetic equation and approximation of it.

  [Degond et al. 2005, 2007, 2010], [Degond and Jin 2005]

Fuzzy sets

[Zadeh 1965]

Let X be a set in the classical sense of generic elements x, such that $X = \{x\}$.

Definition (Fuzzy Set)

A *fuzzy set* A of X is characterized by a *membership function* $h_A(x)$ that associates to every point of X a real number in $[0, 1]$. The value of $h_A(x)$ represents the *grade of membership of* x *in* A. The *support* $\mathrm{Supp}(A)$ of a fuzzy set A is the classical subset of X defined by $\mathrm{Supp}(A) = \{x \in X | h_A(x) \neq 0\}$.

Remark

If the membership function is a characteristic function, then we recover the classical notion of sets.

Properties of fuzzy sets I

[Zadeh 1965]

Definition (Complementary set)

The *complementary set* A^c of a fuzzy set A is defined by its membership function $h_{A^c} = 1 - h_A$.

Definition (Union of fuzzy sets)

The *union* of two fuzzy sets A and B of membership function $h_A(x)$ and $h_B(x)$ is the fuzzy set C, denoted by $C = A \cup B$. It is characterized by its membership function $h_C(x)$ linked with those of A and B by $h_C(x) = \max(h_A(x), h_B(x))$, $\forall x \in X$.

Remark

The union of a fuzzy set with its complementary set is not equal to the initial set, unless the membership functions are characteristic functions: $A \cup A^c \subsetneq X$.

Properties of fuzzy sets II

[Zadeh 1965]

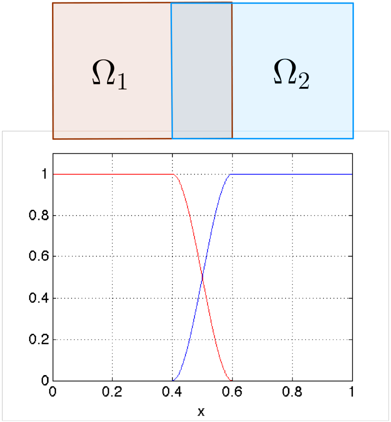
Definition (Algebraic sum of fuzzy sets)

The *algebraic sum* of A and B is denoted by $A + B$ and is defined by the membership function $h_{A+B} = h_A + h_B$. This definition has a meaning only if $h_A(x) + h_B(x) \leq 1$, $\forall x \in X$.

Remark

Note that the algebraic sum has the property that $A + A^c = X$.

Fuzzy Domain Decomposition



We decompose the domain Ω (a set of points) into an algebraic sum of fuzzy domains (fuzzy sets of points) $\Omega_1, \ldots, \Omega_n$ with membership functions $h_1, \ldots, h_n$.

- $\Omega = \Omega_1 + \cdots + \Omega_n$.
- $\sum_{i=1}^{n} h_i(x) = 1, \forall x \in \Omega$.

We say that the family $\{\Omega_i\}_{i=1}^{n}$ is a *Fuzzy Domain Decomposition (FDD)* of Ω.

Fuzzy Domain Decomposition Methods

Definition (FDDM, eFDDM, iFDDM)

A *FDD method* (FDDM) is a numerical method to solve $\mathcal{L}(u) = g$ on Ω based on an FDD of Ω. We will say that an FDDM is explicit (eFDDM) if the membership functions h_i are explicitly known, and implicit otherwise (iFDDM).

Ingredients for a heterogeneous DD method

- Coupling methodology between the approximations
- Criterion to decide which approximation to use where. (Domain of validity of the different approximations)

Advantages of an eFDDM

- The membership functions h_i are used both for implementing the coupling and the criterion.
- The membership functions h_i may be time dependent. (Adaptive decomposition)
- Ease of change of the criterion used.

Assumption

Assumption (eFDDM)

Let u be a function from Ω to $\mathbb{R}$, let $\{\Omega_i\}_{i=1}^n$ be a FDD of Ω and let $u_i := h_i u$ be the restriction of u to Ω_i. Then

$$u = \sum_{i=1}^n u_i \quad \text{and} \quad u' = \sum_{i=1}^n u_i'.$$

And for h_i sufficiently smooth

$$\begin{aligned} u_i &= h_i u, \\ u_i' &= h_i' u + h_i u', \\ u_i'' &= h_i'' u + 2h_i' u' + h_i u'', i = 1, 2. \end{aligned}$$

eFDDM on a model problem

Let $\mathcal{L}$ be a linear differential operator and $\{\Omega_i\}_{i=1,2}$ an FDD defined through $\{h_i\}_{i=1,2}$. Let $\{\mathcal{L}_i\}_{i=1,2}$ be two approximations associated with this FDD. With the eFDDM assumption we have

Derivation of an eFDDM

$$\mathcal{L}(u^*) = g$$

$$\Leftrightarrow \quad h_1\mathcal{L}(u^*) + h_2\mathcal{L}(u^*) = g,$$

$$\rightsquigarrow \quad h_1\mathcal{L}_1(u) + h_2\mathcal{L}_2(u) = g,$$

$$\begin{cases} h_1\mathcal{L}_1(u) = h_1 g \text{ on } \Omega, \\ h_2\mathcal{L}_2(u) = h_2 g \text{ on } \Omega, \end{cases}$$

$$\begin{cases} \widetilde{\mathcal{L}}_1(u_1) = h_1 g + \mathcal{L}_{12}(u_2) \text{ on } \mathrm{Supp}(\Omega_1), \\ \widetilde{\mathcal{L}}_2(u_2) = h_2 g + \mathcal{L}_{21}(u_1) \text{ on } \mathrm{Supp}(\Omega_2), \end{cases}$$

with $\widetilde{\mathcal{L}}_i = \mathcal{L}_i - \mathcal{L}_{i,3-i}, \quad i = 1, 2.$

Advection dominated diffusion

$$\mathcal{L} := \nu\partial_{xx} + a\partial_x \text{ and } g := 0,$$

$\nu > 0, a > 0.$

Approximated operators:

$\mathcal{L}_1 := \mathcal{L} \quad \text{and} \quad \mathcal{L}_2 := a\partial_x$

Approximated problem:

$$\rightsquigarrow \quad h_1\nu u'' + au' = 0,$$

$\mathcal{L}_{12} := \nu(h_1'' + 2h_1'\partial_x) + ah_1',$

$\mathcal{L}_{21} := ah_2',$

Advection dominated diffusion I

We consider for $\nu, a > 0$ the 1D advection diffusion equation

$$\mathcal{L}(u^*) = \nu u^{*\prime\prime} + a u^{*\prime} = 0 \quad \text{on } (0,1),$$
$$u^*(0) = 0, \quad u^*(1) = 1,$$

Approximation

$$\begin{cases} \widetilde{\mathcal{L}}_1(u_1) = h_1 g + \mathcal{L}_{12}(u_2) & \text{on } \mathrm{Supp}(\Omega_1), \\ \widetilde{\mathcal{L}}_2(u_2) = h_2 g + \mathcal{L}_{21}(u_1) & \text{on } \mathrm{Supp}(\Omega_2), \end{cases}$$

with $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_2 = a\partial_x$, $\mathcal{L}_{12} := \nu(h_1'' + 2h_1'\partial_x) + ah_1'$, $\mathcal{L}_{21} := ah_2'$ and $\widetilde{\mathcal{L}}_i = \mathcal{L}_i - \mathcal{L}_{i,3-i}$.

Remark

The boundary conditions of an eFDDM can be easily defined by transferring the boundary conditions on u to u_i using a similar procedure.

Advection dominated diffusion II

- The closed form solution of our model problem

$$\mathcal{L}(u^*) = \nu u^{*\prime\prime} + a u^{*\prime} = 0 \quad \text{on } (0,1), \quad u^*(0) = 0, \quad u^*(1) = 1,$$

  is given by

$$u^*(x) = \frac{e^{-ax/\nu} - 1}{e^{-a/\nu} - 1}.$$

- The analytical solution of our approximation equivalent to

$$h_1 \nu u'' + a u' = 0 \quad \text{on } (0,1), \quad u(0) = 0, \quad u(1) = 1,$$

  is given by

$$u(x) = \begin{cases} \dfrac{\displaystyle\int_0^x \left(e^{-\int_0^y \frac{a}{\nu h_1(z)} dz}\right) dy}{\displaystyle\int_{\mathrm{Supp}(\Omega_1)} \left(e^{-\int_0^y \frac{a}{\nu h_1(z)} dz}\right) dy}, & x \in \mathrm{Supp}(\Omega_1) \\ 1, & \text{otherwise.} \end{cases}$$

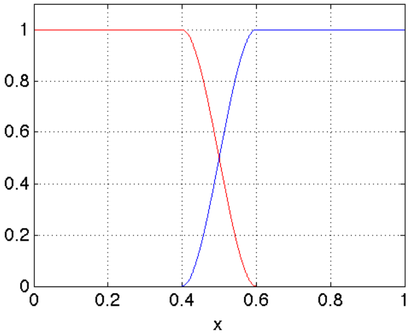
Definition of h_i

In the following, we will use a membership function of the form

$$h_1(x) := \begin{cases} 1, & \text{if } 0 \leq x \leq c_1, \\ h(x), & \text{if } c_1 < x < c_2, \\ 0, & \text{if } c_2 \leq x \leq 1. \end{cases}$$

We also define

$$\delta := c_2 - c_1.$$



Theorem (Approximation Quality [M, Gander 2013])

For the choice of h_1 given above, the relative L^2-error $err_{App}(\frac{\nu}{a}) := \frac{\|u-u^*\|_{L^2(0,1)}}{\|u^*\|_{L^2(0,1)}}$ satisfies when $\frac{\nu}{a} \to 0$ the estimates:

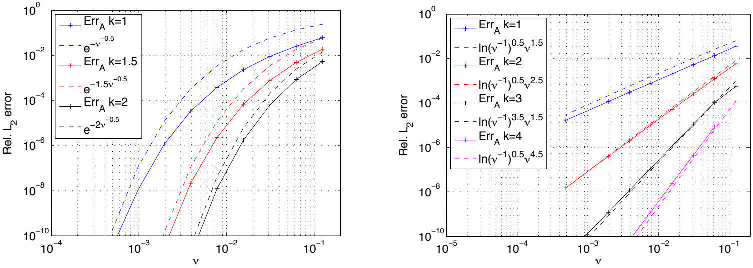
	$c_1 = cst.$, $\delta = cst.$	$c_1 = \kappa\left(\frac{\nu}{a}\right)^{1-\varepsilon}$, $\delta = \kappa'\left(\frac{\nu}{a}\right)^{1-\varepsilon}$	$c_1 = \kappa\frac{\nu}{a}\ln(\frac{a}{\nu})$, $\delta = \kappa'\frac{\nu}{a}$	$c_1 = \kappa\frac{\nu}{a}$, $\delta = \kappa'\frac{\nu}{a}$
$err_{App}(\frac{\nu}{a})$	$O\left(e^{-\frac{ac_1}{\nu}}\right)$	$O\left(e^{-\kappa(\frac{a}{\nu})^{\varepsilon}}\right)$	$O\left(\ln(\frac{a}{\nu})^{0.5}(\frac{\nu}{a})^{\kappa+0.5}\right)$	$O\left((\frac{\nu}{a})^{0.5}\right)$

Here, $\kappa > 0$, $\kappa' \geq 0$ are constants, and $0 < \varepsilon \leq 1$.

Sketch of the proof

1. Construct v_1 and v_2 such that $v_1 \leq u \leq v_2$ using the fact that $0 < h \leq 1$.
2. Estimate $e_i^2 := \|\frac{v_i - u^*}{u^*}\|^2_{L^2(0,1)}$. This implies $\text{err}_{\text{App}}(\frac{\nu}{a}) \leq \max_i e_i$.
3. Conclude by studying the different cases.

This theorem shows that the approximation quality of the method is similar to the best known coupling methods for this kind of problem, namely the one based on the factorization of the operator. [Gander, Martin 2012]



(a) Case 2: $c_1 = k\frac{\nu}{a}^{1-\varepsilon}$, $\delta = \frac{\nu}{a}^{1-\varepsilon}$, with $a = 1$, $\varepsilon = 0.5$ and $k = 1, 1.5, 2$.

(b) Case 3: $c_1 = k\frac{\nu}{a}\ln(\frac{a}{\nu})$, $\delta = \frac{\nu}{a}$, with $a = 1$ and $k = 1, 2, 3, 4$.

Figure: Results for the cases 2 and 3 of the Theorem where we refined the grid keeping $n\nu$ constant. We see that the curves follow the theoretical predictions.

Conclusions

- We have motivated and presented the new class of Fuzzy Domain Decomposition Methods.
- We have shown on an example that the method works well and that the approximation quality is very good both theoretically and experimentally.

Future work

- Generalize the former results to more complicated 1D system
- Generalize the method to higher dimensional problems
- Add time dependency in the problem and in the FDD
- Develop good criterion for the domain of validity of the relevant approximations
- Develop an eFDDM for radiative transfer of neutrinos in core-collapse supernovae

Thank you
for your attention

Step 1:

- With h_1 as in the theorem, we can express the function u as

$$u(x) = \begin{cases} \dfrac{1-e^{-\frac{ax}{\nu}}}{1-e^{-\frac{ac_1}{\nu}}\left(1-\frac{a}{\nu}\int_{c_1}^{c_2} e^{-\frac{a}{\nu}\int_{c_1}^{y} h^{-1}(z)dz}dy\right)}, & \text{if } 0 \leq x \leq c_1, \\ \dfrac{1-e^{-\frac{ac_1}{\nu}}\left(1-\frac{a}{\nu}\int_{c_1}^{x} e^{-\frac{a}{\nu}\int_{c_1}^{y} h^{-1}(z)dz}dy\right)}{1-e^{-\frac{ac_1}{\nu}}\left(1-\frac{a}{\nu}\int_{c_1}^{c_2} e^{-\frac{a}{\nu}\int_{c_1}^{y} h^{-1}(z)dz}dy\right)}, & \text{if } c_1 < x < c_2, \\ 1, & \text{if } c_2 \leq x \leq 1. \end{cases}$$

- Using the fact that $0 < h(z) \leq 1$, we have for $c_1 < x < c_2$ the estimate

$$1 - e^{-\frac{ac_1}{\nu}} < 1 - e^{-\frac{ac_1}{\nu}}\left(1 - \frac{a}{\nu}\int_{c_1}^{x} e^{-\frac{a}{\nu}\int_{c_1}^{y} h^{-1}(z)dz}dy\right) \leq 1 - e^{-\frac{ax}{\nu}}.$$

- Using this estimate, we define v_i, $i = 1, 2$ as

$$\left.\begin{array}{ll} \text{if } 0 \leq x \leq c_1, & \frac{1-e^{-\frac{ax}{\nu}}}{1-e^{-\frac{ac_2}{\nu}}} \\ \text{if } c_1 < x < c_2, & \frac{1-e^{-\frac{ac_1}{\nu}}}{1-e^{-\frac{ac_2}{\nu}}} \\ \text{if } c_2 \leq x \leq 1, & 1 \end{array}\right\} =: v_1(x) \leq u(x) \leq v_2(x) := \begin{cases} \frac{1-e^{-\frac{ax}{\nu}}}{1-e^{-\frac{ac_1}{\nu}}} & \text{if } 0 \leq x \leq c_1, \\ \frac{1-e^{-\frac{ax}{\nu}}}{1-e^{-\frac{ac_1}{\nu}}} & \text{if } c_1 < x < c_2, \\ 1 & \text{if } c_2 \leq x \leq 1. \end{cases}$$

Step 2:

We now compute the relative L^2-errors for v_i, $i = 1, 2$. With $e_i := \|\frac{v_i - u^*}{u^*}\|_{L^2(0,1)}$, we have

$$e_1^2 = I_1(1,2) + I_2 + I_3 \quad \text{and} \quad e_2^2 = I_1(2,1) + I_3,$$

where the different terms are integrals of the form $\int (\frac{v_i}{u} - 1)^2 dx$,

- $I_1(i,j) := \int_0^{c_i} \left(\frac{1 - e^{-\frac{a}{\nu}}}{1 - e^{-\frac{ac_j}{\nu}}} - 1 \right)^2 dx = c_i \left(\frac{1 - e^{-\frac{a}{\nu}}}{1 - e^{-\frac{ac_j}{\nu}}} - 1 \right)^2 = O\left(c_i (\frac{\nu}{a}) e^{-\frac{2ac_j(\frac{\nu}{a})}{\nu}} \right)$,
- $I_2 := \int_{c_1}^{c_2} \left[\frac{(1 - e^{-\frac{ac_1}{\nu}})(1 - e^{-\frac{a}{\nu}})}{(1 - e^{-\frac{ac_2}{\nu}})(1 - e^{-\frac{ax}{\nu}})} - 1 \right]^2 dx \leq \delta \max_{i=1,2} \left(\left[\frac{(1 - e^{-\frac{ac_1}{\nu}})(1 - e^{-\frac{a}{\nu}})}{(1 - e^{-\frac{ac_2}{\nu}})(1 - e^{-\frac{ac_i}{\nu}})} - 1 \right]^2 \right) = O\left(\delta (\frac{\nu}{a}) e^{-\frac{2ac_1(\frac{\nu}{a})}{\nu}} \right)$,
- $I_3 := \int_{c_2}^{1} \left(\frac{1 - e^{-\frac{a}{\nu}}}{1 - e^{-\frac{ax}{\nu}}} - 1 \right)^2 dx = \int_{c_2}^{1} \left[\sum_{k=1}^{\infty} e^{-\frac{kax}{\nu}} (1 - e^{-\frac{a}{\nu}}) - e^{-\frac{a}{\nu}} \right]^2 dx = O\left(\frac{\nu}{a} e^{-\frac{2ac_2(\frac{\nu}{a})}{\nu}} \right)$.

Estimation of I_i

- As $e^{-\frac{ac_i}{\nu}} < 1$ and $e^{-\frac{ax}{\nu}} < 1$, we can use geometric series to obtain estimates of the different integrals.
- Taking only the leading term gives the result for $I_1(i,j)$ and I_3.
- For I_3, the leading term under the integration is $e^{-\frac{ax}{\nu}}$, because $x \leq 1$.
- For I_2 we also used the monotonicity of the exponential to obtain the bound and then, use once again a geometric series to conclude.
- In the order notation, we have specified the possible dependence of c_i and δ on the parameter $\frac{\nu}{a}$.

Step 3:

We now need to distinguish the different cases in order to complete the proof. Using the estimates for $I_1(i,j)$, I_2 and I_3, we can compute the results shown in the following table.

	$c_1 = \text{cst.}$, $\delta = \text{cst.}$	$c_1 = \kappa\left(\frac{\nu}{a}\right)^{1-\varepsilon}$, $\delta = \kappa'\left(\frac{\nu}{a}\right)^{1-\varepsilon}$	$c_1 = \kappa\frac{\nu}{a}\ln(\frac{a}{\nu})$, $\delta = \kappa'\frac{\nu}{a}$	$c_1 = \kappa\frac{\nu}{a}$, $\delta = \kappa'\frac{\nu}{a}$
$I_1(1,2)$	$O(e^{-\frac{2ac_2}{\nu}})$	$O\left(e^{-2(\kappa+\kappa')(\frac{a}{\nu})^{\varepsilon}}\right)$	$O(\ln(\frac{a}{\nu})(\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$
$I_1(2,1)$	$O(e^{-\frac{2ac_1}{\nu}})$	$O\left(e^{-2\kappa(\frac{a}{\nu})^{\varepsilon}}\right)$	$O(\ln(\frac{a}{\nu})(\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$
I_2	$O(e^{-\frac{2ac_1}{\nu}})$	$O\left(e^{-2\kappa(\frac{a}{\nu})^{\varepsilon}}\right)$	$O((\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$
I_3	$O(e^{-\frac{2ac_2}{\nu}})$	$O\left(e^{-2(\kappa+\kappa')(\frac{a}{\nu})^{\varepsilon}}\right)$	$O((\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$
e_1^2	$O(e^{-\frac{2ac_1}{\nu}})$	$O\left(e^{-2\kappa(\frac{a}{\nu})^{\varepsilon}}\right)$	$O(\ln(\frac{a}{\nu})(\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$
e_2^2	$O(e^{-\frac{2ac_1}{\nu}})$	$O\left(e^{-2\kappa(\frac{a}{\nu})^{\varepsilon}}\right)$	$O(\ln(\frac{a}{\nu})(\frac{\nu}{a})^{2\kappa+1})$	$O(\frac{\nu}{a})$

Table: Table of the order of the different integrals I_j.

Finally, we conclude with $err_{App} \leq \max_{i=1,2} e_i$. □