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A deluxe FETI-DP algorithm for a hybrid staggered DG formulation of H(curl) in two dimensions

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Contents

- Hybrid staggered DG formulation
- A FETI-DP algorithm with deluxe scaling
- Analysis of condition numbers
- Numerical results

Find $\vec{u} \in H_0(\text{curl}, \Omega)$ such that

$$\nabla \times (\alpha(x)\nabla \times \vec{u}(x)) + \beta(x)\vec{u}(x) = \vec{f}(x) \quad \forall x \in \Omega \subset \mathbb{R}^2,$$

where

$$H_0(\text{curl}, \Omega) = \{\vec{v} \in [L^2(\Omega)]^2 \;:\; \nabla\times\vec{v} \in L^2(\Omega),\; \vec{v}\cdot\vec{t} = 0 \;\text{ on } \partial\Omega\},$$

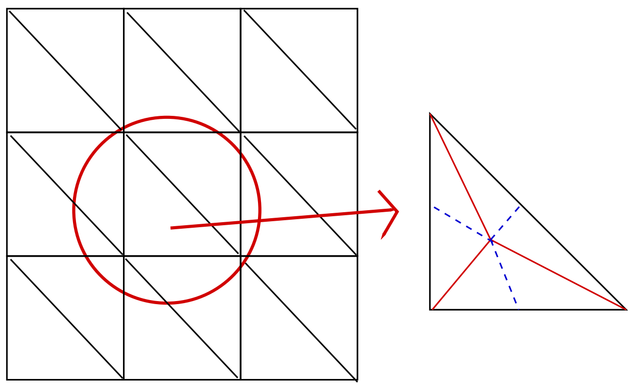
$\alpha(x)$, $\beta(x)$ (> 0) can be discontinuous.

First order system:

$$\begin{aligned} q &= \alpha(x)\nabla \times \vec{u}(x) \\ \nabla \times q + \beta\vec{u}(x) &= \vec{f}(x) \end{aligned}$$

Staggered DG formulation

(by Chung and Enquist (2006,2009))



initial triangulation and subdivision

$\mathcal{T}$: initial triangulation
$\mathcal{T}_s$: subdivided triangulation
$\mathcal{F}$: black edges
$\mathcal{F}_u$: red edges
$\mathcal{F}_q$: blue edges

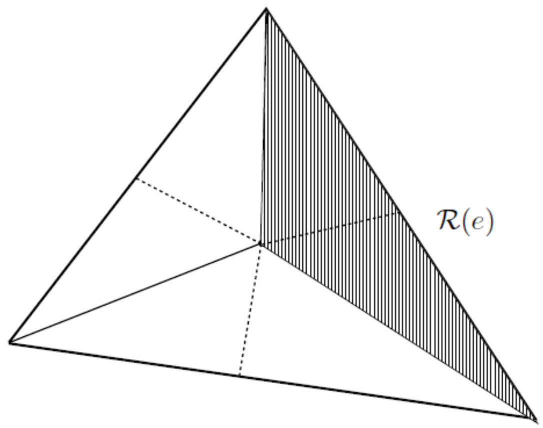
$$\mathcal{S}^h = \{\psi \, : \, \psi|_\tau \in P^k(\tau) \, \forall \tau \in \mathcal{T}_s, \; [\psi]|_e = 0 \; \forall e \in \mathcal{F}_q\}.$$

$$\mathcal{V}^h = \{\vec{v} \, : \, \vec{v}|_\tau \in [P^k(\tau)]^2 \, \forall \tau \in \mathcal{T}_s, \; [\vec{v} \cdot \vec{t}]|_e = 0 \; \forall e \in \mathcal{F}_u\}.$$

Note: Both are decoupled across edges in $\mathcal{F}$.

Weak formulation

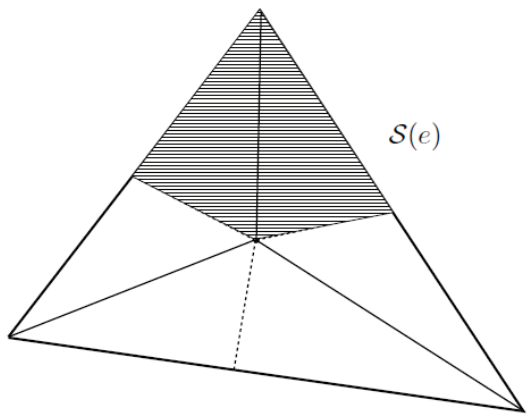
ψ in $\mathcal{S}_h$ is locally H^1-conforming in $\mathcal{R}(e)$ for e in $\mathcal{F}_q$:



$$q = \alpha(x)\nabla \times \vec{u}(x)$$

$$\int_{\mathcal{R}(e)} \alpha^{-1} q\psi \, dx - \int_{\mathcal{R}(e)} \vec{u}\cdot(\nabla \times \psi) \, dx - \int_{\partial\mathcal{R}(e)} (\vec{u}\cdot\vec{t}_\tau)\psi \, d\sigma = 0.$$

$\vec{v}$ in $\mathcal{V}_h$ are locally H-curl conforming in $\mathcal{S}(e)$ for e in $\mathcal{F}_u$:



$$\nabla \times q + \beta \vec{u}(x) = \vec{f}(x)$$

$$\int_{\mathcal{S}(e)} q(\nabla \times \vec{v})dx - \int_{\partial\mathcal{S}(e)} q(\vec{v}\cdot\vec{t}_\tau)d\sigma + \int_{\mathcal{S}(e)} \beta\vec{u}\cdot\vec{v}dx = \int_{\mathcal{S}(e)} \vec{f}\cdot\vec{v}dx.$$

Hybridization: We will approximate q on e in $\mathcal{F} \bigcap \partial\mathcal{S}(e)$ by introducing an additional unknown λ.

Hybridization

- For edges e in $\mathcal{F}$, we approximated q with λ.
- In addition we enforce continuity of $\vec{u} \cdot \vec{t}$,

$$[\vec{u} \cdot \vec{t}] = 0, \quad \forall e \in \mathcal{F}$$

weakly by introducing Lagrange multipliers.

Resulting algebraic system

$$\begin{aligned} M_{\alpha^{-1}} q - B^T u &= 0 \\ Bq + N_\beta u + J^T \lambda &= 0 \\ Ju &= 0. \end{aligned}$$

- **q and u are decoupled across edges in the initial triangulation.**
- No penalty terms due to the hybridization and staggered continuity.
- **λ is defined on edges in the initial triangulation.** It approximates q and can be interpreted as the Lagrange multipliers to $Ju = 0$.

Elimination of q and u gives

$$J(BM_{\alpha^{-1}}^{-1} B^T + N_\beta)^{-1} J^T \lambda = d.$$

We will develop **a FETI-DP algorithm** for

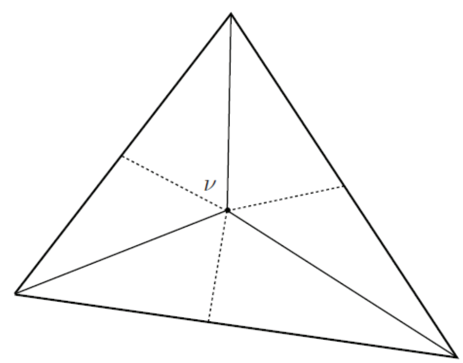
$$J(BM_\alpha^{-1}B^T + N_\beta)^{-1}J^T\lambda = d.$$

- Nonoverlapping partition

$$\Omega = \bigcup_i \Omega_i,$$

  Ω_i: a connected union of triangles in the **initial** triangulation $\mathcal{T}$.

- Assume that $\alpha(x)$ and $\beta(x)$ are positive constants in each subdomain Ω_i.



For a given $u_{\partial\tau}$ on $\partial\tau$ find $(q_\tau^{\mathcal{D}}, u_\tau^{\mathcal{D}})$ such that

$$\begin{aligned} \alpha_\tau^{-1} M_\tau q_\tau^{\mathcal{D}} - B_\tau^T u_\tau^{\mathcal{D}} &= 0 \\ B_\tau q_\tau^{\mathcal{D}} + \beta_\tau N_\tau u_\tau^{\mathcal{D}} &= 0 \\ u_\tau^{\mathcal{D}} \cdot \vec{t} &= u_{\partial\tau} \quad \text{on } \partial\tau. \end{aligned}$$

For a given $\lambda_{\partial\tau}$ on $\partial\tau$ find $(q_\tau^{\mathcal{N}}, u_\tau^{\mathcal{N}})$ such that

$$\begin{aligned} \alpha_\tau{}^{-1} M_\tau q_\tau^{\mathcal{N}} - B_\tau^T u_\tau^{\mathcal{N}} &= 0 \\ B_\tau q_\tau^{\mathcal{N}} + \beta_\tau N_\tau u_\tau^{\mathcal{N}} - J_\tau^T \lambda_{\partial\tau} &= 0. \end{aligned}$$

Assembly of Neumann problems

Let

$$A_\tau = \alpha_\tau B_\tau M_\tau^{-1} B_\tau^T + \beta_\tau N_\tau.$$

We introduce **the Schur complement of A_τ**:

$$\mathcal{D}_{\partial\tau} = A_{\tau,BB} - A_{\tau,BI} A_{\tau,II}^{-1} A_{\tau,IB}$$

$$(I: \text{ interior to } \tau \quad B: \text{ dofs on } \partial\tau)$$

For **the Neumann problem**, we observe that

$$u_\tau^{\mathcal{N}}|_{\partial\tau} = \mathcal{D}_{\partial\tau}^{-1} J_\tau^T \lambda_{\partial\tau}$$

and thus our algebric system on λ can be written into

$$\sum_{\tau\in\mathcal{T}} J_\tau \mathcal{D}_{\partial\tau}^{-1} J_\tau^T \lambda = d.$$

We introduce

$$\mathcal{N}_i = \sum_{\tau \in \Omega_i \bigcap \mathcal{T}} J_\tau (\mathcal{D}_{\partial\tau})^{-1} J_\tau^T, \quad \mathcal{D}_i = \sum_{\tau \in \Omega_i \bigcap \mathcal{T}} R_\tau^T \mathcal{D}_{\partial\tau} R_\tau. \ (*)$$

The Schur complements of $\mathcal{N}_i$ and $\mathcal{D}_i$,

$$S_i^{\mathcal{N}} = \mathcal{N}_{i,\Gamma\Gamma} - \mathcal{N}_{i,\Gamma I} \mathcal{N}_{i,II}^{-1} \mathcal{N}_{i,I\Gamma}, \quad S_i^{\mathcal{D}} = \mathcal{D}_{i,\Gamma\Gamma} - \mathcal{D}_{i,\Gamma I} \mathcal{D}_{i,II}^{-1} \mathcal{D}_{i,I\Gamma}.$$

(I: interior to Ω_i, Γ: dofs on $\partial\Omega_i$)

By the relation in (*),

$$S_i^{\mathcal{N}} = (S_i^{\mathcal{D}})^{-1}.$$

Interface problem

After eliminating λ_I interior to each subdomain, we obtain the interface problem on λ_Γ,

$$\sum_i J_{\Gamma,i} S_i^{\mathcal{N}} J_{\Gamma,i}^T \lambda_\Gamma = d_\Gamma.$$

- $S_i^{\mathcal{N}}$: subdomain problem assembled by Neumann problem matrix $\mathcal{D}_{\partial\tau}^{-1}$
- $S_i^{\mathcal{D}}$: subdomain problem assembled by Dirichlet problem matrix $\mathcal{D}_{\partial\tau}$
- $$S_i^{\mathcal{N}} = (S_i^{\mathcal{D}})^{-1}$$

Change of unknowns

For $\Gamma_{ij} = \partial\Omega_i \bigcap \partial\Omega_j$, we consider a change of basis on $u_{\Gamma_{ij}}$ such that

$$u_{\Gamma_{ij}} = T_{ij} \begin{pmatrix} \widehat{u}_{\Pi,ij} \\ \widehat{u}_{\Delta,ij} \end{pmatrix},$$

$$\widehat{u}_{\Pi,ij} = \frac{\int_{\Gamma_{ij}} \vec{u} \cdot \vec{t}\, ds}{\int_{\Gamma_{ij}} 1\, ds}, \quad \widehat{u}_{\Delta,ij} : \text{ average free.}$$

For $\lambda_{\Gamma_{ij}}$, we introduce the primal unknowns

$$\lambda_{\Gamma_{ij}} = (T_{ij}^T)^{-1} \begin{pmatrix} \widehat{\lambda}_{\Pi,ij} \\ \widehat{\lambda}_{\Delta,ij} \end{pmatrix}.$$

Change of unknowns

Let

$$T = \text{diag}_{ij} T_{ij}, \quad T_i = \text{diag}_j T_{ij}.$$

By the change of unknowns using $(T^T)^{-1}$,
(Note that $S_i^{\mathcal{N}} = (S_i^{\mathcal{D}})^{-1}$)

$$T^{-1} \sum_i J_{\Gamma,i} S_i^{\mathcal{N}} J_{\Gamma,i}^T (T^T)^{-1} = \sum_i J_{\Gamma,i} T_i^{-1} S_i^{\mathcal{N}} (T_i^T)^{-1} J_{\Gamma,i}^T$$
$$= \sum_i J_{\Gamma,i} (T_i^T S_i^{\mathcal{D}} T_i)^{-1} J_{\Gamma,i}^T,$$

the identity relation is thus preserved

$$\sum_i J_{\Gamma,i} \widehat{S}_i^{\mathcal{N}} J_{\Gamma,i}^T = \sum_i J_{\Gamma,i} (\widehat{S}_i^{\mathcal{D}})^{-1} J_{\Gamma,i}^T$$

for $\widehat{S}_i^{\mathcal{N}} = T_i^{-1} S_i^{\mathcal{N}} (T_i^T)^{-1}$, $\widehat{S}_i^{\mathcal{D}} = T_i^T S_i^{\mathcal{D}} T_i$.

Change of unknowns

- Interface problem

$$\sum_i J_{\Gamma,i} S_i^{\mathcal{N}} J_{\Gamma,i}^T \lambda_\Gamma = d_\Gamma$$

- Change of unknowns using $(T^T)^{-1}$

$$\sum_i J_{\Gamma,i} \widehat{S}_i^{\mathcal{N}} J_{\Gamma,i}^T \widehat{\lambda}_\Gamma = \widehat{d}_\Gamma$$

- Inverse property

$$\widehat{S}_i^{\mathcal{N}} = (\widehat{S}_i^{\mathcal{D}})^{-1}$$

for the transformed local problems

$$\widehat{S}_i^{\mathcal{N}} = T_i^{-1} S_i^{\mathcal{N}} (T_i^T)^{-1}, \quad \widehat{S}_i^{\mathcal{D}} = T_i^T S_i^{\mathcal{D}} T_i.$$

FETI-DP algorithm

Let $\widehat{\lambda}_\Gamma = \begin{pmatrix} \lambda_\Pi \\ \lambda_\Delta \end{pmatrix}$. We order the interface problem into

$$\sum_i J_{\Gamma,i} \widehat{S}_i^{\mathcal{N}} J_{\Gamma,i}^T \widehat{\lambda}_\Gamma = \begin{pmatrix} F_{\Pi\Pi} & F_{\Pi\Delta} \\ F_{\Delta\Pi} & F_{\Delta\Delta} \end{pmatrix} \begin{pmatrix} \lambda_\Pi \\ \lambda_\Delta \end{pmatrix}$$

and eliminate λ_Π to obtain

$$F_{DP} \lambda_\Delta = d_\Delta,$$

where

$$F_{DP} = F_{\Delta\Delta} - F_{\Delta\Pi} F_{\Pi\Pi}^{-1} F_{\Pi\Delta}.$$

Note that $F_{\Pi\Pi}$ is the coarse component of our algorithm.

We solve

$$F_{DP}\lambda_\Delta = d_\Delta$$

with a preconditioner M^{-1}.

For the ordered matrix $\widehat{S}_i^{\mathcal{N}} = \begin{pmatrix} \widehat{S}_{\Pi\Pi}^{\mathcal{N},i} & \widehat{S}_{\Pi\Delta}^{\mathcal{N},i} \\ \widehat{S}_{\Delta\Pi}^{\mathcal{N},i} & \widehat{S}_{\Delta\Delta}^{\mathcal{N},i} \end{pmatrix}$, we have

$$F_{\Pi\Pi} = \sum_i J_{\Pi,i} \widehat{S}_{\Pi\Pi}^{\mathcal{N},i} J_{\Pi,i}^T$$

$$F_{\Pi\Delta} = \sum_i J_{\Pi,i} \widehat{S}_{\Pi\Delta}^{\mathcal{N},i} J_{\Delta,i}^T \qquad F_{\Delta\Pi} = F_{\Pi\Delta}^T$$

$$F_{\Delta\Delta} = \sum_i J_{\Delta,i} \widehat{S}_{\Delta\Delta}^{\mathcal{N},i} J_{\Delta,i}^T.$$

The calculation of $F_{DP}\lambda_\Delta$ can be done by solving local problems and one coarse problem.

From the ordered matrix, $\widehat{S}_i^{\mathcal{D}} = \begin{pmatrix} \widehat{S}_{\Pi\Pi}^{\mathcal{D},i} & \widehat{S}_{\Pi\Delta}^{\mathcal{D},i} \\ \widehat{S}_{\Delta\Pi}^{\mathcal{D},i} & \widehat{S}_{\Delta\Delta}^{\mathcal{D},i} \end{pmatrix}$, we define the weight factor $D_{\Delta,i}$,
(Dohrmann, Widlund (2013) DD20 Proceedings)

$$D_{\Delta,i}|_F = (S_F^{(i)} + S_F^{(j)})^{-1} S_F^{(j)}, \quad D_{\Delta,j}|_F = (S_F^{(i)} + S_F^{(j)})^{-1} S_F^{(i)},$$

with $S_F^{(i)}$ is the block matix of $\widehat{S}_{\Delta\Delta}^{\mathcal{D},i}$ to the unknowns in $F = \partial\Omega_i \bigcap \partial\Omega_j$.

The preconditioner consists of solving local problems and then weighted assembly,

$$M^{-1} = \sum_i J_{\Delta,i}^T D_{\Delta,i}^T \widehat{S}_{\Delta\Delta}^{\mathcal{D},i} D_{\Delta,i} J_{\Delta,i}.$$

Estimate of condition numbers

For $u_\Gamma = \begin{pmatrix} u_\Pi \\ u_\Delta \end{pmatrix}$, we introduce a partially coupled space $\widetilde{W}$ which consists of u_Γ coupled at the primal unknowns u_Π. We subassemble local matrices $\widehat{S}^{\mathcal{D},i}$ at the primal unknowns to obtain a matrix defined on $\widetilde{W}$,

$$\widetilde{S}^{\mathcal{D}} = \begin{pmatrix} \widetilde{S}^{\mathcal{D}}_{\Pi\Pi} & \widetilde{S}^{\mathcal{D}}_{\Pi\Delta} \\ \widetilde{S}^{\mathcal{D}}_{\Delta\Pi} & \widetilde{S}^{\mathcal{D}}_{\Delta\Delta} \end{pmatrix}, \quad \widetilde{D} = \begin{pmatrix} 0 & 0 \\ 0 & D_{\Delta\Delta} \end{pmatrix}, \quad \widetilde{J}^T = \begin{pmatrix} 0 \\ J^T_\Delta \end{pmatrix}.$$

Using them, we obtain

$$F_{DP} = \widetilde{J}(\widetilde{S}^{\mathcal{D}})^{-1}\widetilde{J}^T$$

and

$$M^{-1} = \widetilde{J}\widetilde{D}^T\widetilde{S}^{\mathcal{D}}\widetilde{D}\widetilde{J}^T.$$

Estimate of condition numbers

- Using the representation

$$F_{DP} = \widetilde{J}(\widetilde{S}^{\mathcal{D}})^{-1}\widetilde{J}^T, \quad M^{-1} = \widetilde{J}\widetilde{D}^T\widetilde{S}^{\mathcal{D}}\widetilde{D}\widetilde{J}^T$$

and following the standard FETI-DP analysis we obtain the lower bound bounded below by one.

- For the upper bound, we need to estimate

$$\|P_D\|^2_{\widetilde{S}^{\mathcal{D}}}$$

where

$$P_D = \widetilde{D}\widetilde{J}\widetilde{J}^T.$$

Lemma 1. Using the deluxe scaling, we obtain

$$\langle \widetilde{S}^{\mathcal{D}} P_D \widetilde{w}, P_D \widetilde{w} \rangle \leq C \sum_i \sum_{F \subset \Omega_i} \langle S_F^{(i)} w_{\Delta,F}^{(i)}, w_{\Delta,F}^{(i)} \rangle.$$

Lemma 2. (Oh, Widlund, Dohrmann (2013)) For $w^{(i)}$ we obtain that

$$\langle S_F^{(i)} w_{\Delta,F}^{(i)}, w_{\Delta,F}^{(i)} \rangle \leq C \left(1 + \log \frac{H}{h}\right)^2 \langle \widehat{S}^{\mathcal{D},i} w^{(i)}, w^{(i)} \rangle,$$

where C is independent of α_i and β_i.
By the above two lemmas, we obtain that

$$\|P_D\|^2_{\widetilde{S}^{\mathcal{D}}} \leq C \left(1 + \log \frac{H}{h}\right)^2.$$

Model problem

- Ω: a unit square domain

- N_d: number of subdomains
  Note: uniform subdomain partitions

- H/h: numer of triangles across each subdomain

- $\mathcal{S}^h, \mathcal{V}^h$: $(k = 0)$ piecewise constant,
  $(k = 1)$ piecewise linear

- Stop condition: relative residual norm $(< 10^{-6})$

	$\mathcal{V}^h$ with $k = 0$				
H/h	Iter	$\lambda_{\min}$	$\lambda_{\max}$	$\lVert\vec{u} - \vec{u}_h\rVert_0$	order
1	6	1.00	1.22	6.4163e-1	
2	8	1.00	1.62	3.2335e-1	0.99
4	9	1.00	2.18	1.6169e-1	1.00
8	11	1.00	2.88	8.0745e-2	1.00
16	12	1.00	3.71	4.0333e-2	1.00

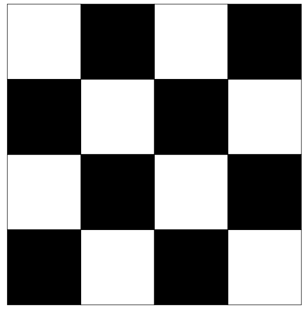
	$\mathcal{V}^h$ with $k = 1$				
H/h	Iter	$\lambda_{\min}$	$\lambda_{\max}$	$\lVert\vec{u} - \vec{u}_h\rVert_0$	order
1	8	1.00	1.81	2.1260e-1	
2	10	1.00	2.47	5.7665e-2	1.88
4	11	1.00	3.26	1.4978e-2	1.94
8	13	1.00	4.17	3.8133e-3	1.97
16	16	1.00	5.63	9.6176e-4	1.99

Model with an exact solution and $\alpha = \beta = 1$

	$\mathcal{V}^h$ with $k = 0$				
N_d	Iter	$\lambda_{\min}$	$\lambda_{\max}$	$\Vert\vec{u} - \vec{u}_h\Vert_0$	order
4^2	9	1.00	2.18	1.6169e-1	
8^2	13	1.00	2.58	8.0745e-2	1.00
16^2	13	1.00	2.73	4.0333e-2	1.00
32^2	14	1.00	2.77	2.0154e-2	1.00

	$\mathcal{V}^h$ with $k = 1$				
N_d	Iter	$\lambda_{\min}$	$\lambda_{\max}$	$\Vert\vec{u} - \vec{u}_h\Vert_0$	order
4^2	11	1.00	3.26	1.4978e-2	
8^2	17	1.00	3.89	3.8133e-3	1.97
16^2	18	1.00	4.10	9.6176e-4	1.99
32^2	19	1.00	4.18	2.4148e-5	1.99

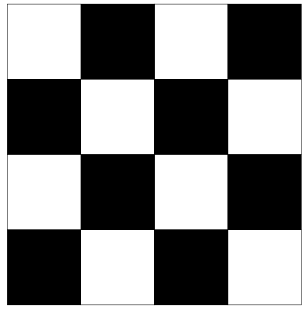
Model with discontinuous $\alpha(x)$



black: $(\alpha_b, \beta_b) = (\alpha_i, 1)$
white: $(\alpha_w, \beta_w) = (1, 1)$
$\mathcal{V}^h$ with $k = 1$

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
α_i	Iter	Cond	Iter	Cond	Iter	Cond	Iter	Cond
10^{-3}	13	2.16	16	2.87	18	3.75	20	4.77
10^{-2}	13	2.46	16	3.26	18	4.20	22	5.27
10^{-1}	13	2.63	15	3.48	17	4.46	20	5.58
1	12	2.66	14	3.51	16	4.50	20	5.63
10^{1}	12	2.66	14	3.51	16	4.51	20	5.63
10^{2}	12	2.66	15	3.52	16	4.51	20	5.63
10^{3}	12	2.66	15	3.52	16	4.51	21	5.63

Model with discontinuous $\beta(x)$



black: $(\alpha_b, \beta_b) = (1, \beta_i)$
white: $(\alpha_w, \beta_w) = (1, 1)$
$\mathcal{V}^h$ with $k = 1$

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
β_i	Iter	Cond	Iter	Cond	Iter	Cond	Iter	Cond
10^{-3}	5	1.01	5	1.01	6	1.02	6	1.02
10^{-2}	7	1.07	7	1.11	8	1.16	9	1.21
10^{-1}	10	1.61	11	1.92	13	2.27	15	2.68
1	12	2.66	14	3.51	16	4.50	20	5.63
10^{1}	9	1.57	11	1.87	12	2.22	13	2.62
10^{2}	5	1.07	6	1.10	7	1.15	8	1.20
10^{3}	4	1.01	4	1.02	5	1.02	5	1.03

Model with discontinuous $\alpha(x), \beta(x)$

Contents
DG formulation
FETI-DP Method
Analysis
▷ Numerical results
Summary

- $N_d = 4^2$, $\mathcal{V}^h$ with $k = 1$
- $\alpha_i = 10^{r_i}, \beta_i = 10^{s_i}$ by choosing r_i and s_i randomly from $(-3, 3)$

	$H/h = 2$		$H/h = 4$		$H/h = 8$		$H/h = 16$	
	Iter	Cond	Iter	Cond	Iter	Cond	Iter	Cond
Set 1	12	2.35	14	3.05	16	3.86	23	4.79
Set 2	12	2.10	14	2.69	16	3.38	21	4.17
Set 3	13	2.48	17	3.30	18	4.26	26	5.37
Set 4	12	1.97	14	2.50	15	3.12	19	3.83
Set 5	13	2.56	15	3.38	17	4.34	21	5.44

- A new hybrid staggered DG formulation with the optimal error bound
- A new FETI-DP algorithm for resulting equation on λ defined on the triangle interfaces
- Local problems are assembled by Dirichlet and Neumann problems at the triangle level.
- **Coarse problem** by the change of unknowns on λ
- **Preconditioner with the deluxe scaling** is robust to the jump of two parameters.
- **The algorithm and analysis** can be applied to any algebraic system arising from **hybridizable DG methods.**

The end

Thank you!