Domain decomposition methods with overlapping subdomains for the time-dependent problems

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Content

Introduction

Domain decomposition methods are used for the numerical solution of boundary value problems for partial differential equations on parallel computers.
Most fully in the theory of domain decomposition methods are presented for stationary problems[1,2].
Computational algorithms with overlap and without overlap of the subdomains are used in synchronous (sequential) and asynchronous (parallel) algorithms.

[1] A. Quarteroni and A. Valli, Domain decomposition methods for partial differential equations, Numerical Mathematics and Scientific Computation. Oxford: Clarendon Press. xv, 360 p., 1999.

[2] A. Toselli and O. Widlund, Domain decomposition methods – algorithms and theory, Springer Series in Computational Mathematics 34. Berlin: Springer. xv, 450 p., 2005.

Domain decomposition methods for unsteady problems are based on two approaches[3].

1. For approximate solution of time-dependent problems we use ordinary implicit approximation in time. Domain decomposition methods applied to solving the discrete problem on the new time level. The number of iterations in the optimal iterative methods for domain decomposition does not depend on the discretization steps in time and space.
2. It is notiterative domain decomposition algorithms are constructed for nonstationary problems. We construct a special scheme of splitting into subdomains (region-additive schemes).

[3] A. Samarskii, P. Matus and P. Vabishchevich, Difference schemes with operator factors, Kluwer Academic Publishers, 2002.

Regionally additive scheme

A domain decomposition scheme is defined by a decomposition of the computational domain and by defining the splitting of the operator. To construct the decomposition operators when solving BVP for PDEs, it is convenient to use a partition of unity for the computational domain.
In the overlapping DD methods, a function is associated with each subdomain, and this function takes value between zero and one.
In the extreme case, the width of the overlap of the subdomains is equal to the space discretization step. In this case the regionally-additive schemes can be interpreted as non-overlapping domain decomposition schemes, where the exchange is achieved by setting proper boundary conditions for each of the subdomain.

The main issues

Domain decomposition methods for unsteady problems:

- Decomposition of domain;
- Operators of decomposition;
- Splitting scheme;
- Study of convergence;
- The computational implementation.

Illustrative example

The boundary value problem for one-dimensional parabolic equation:

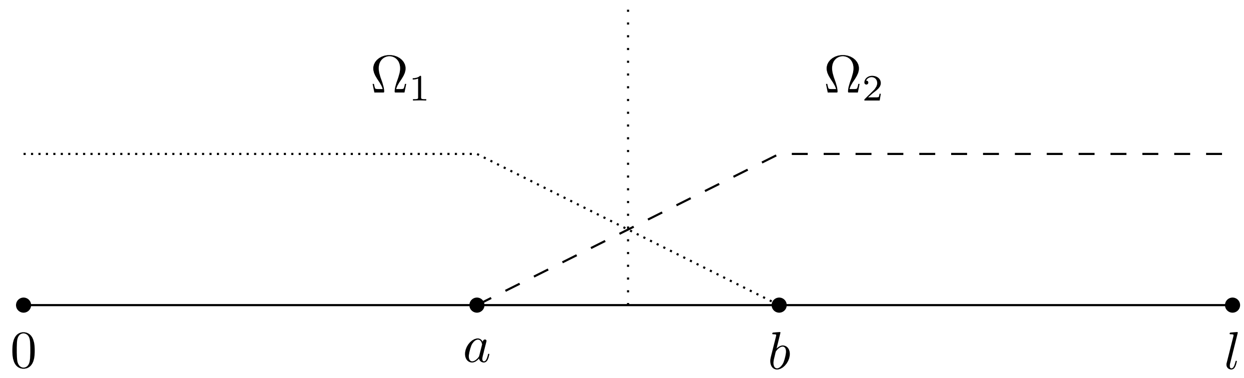
$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < l, \quad 0 < t \le T.$$

$$u(0,t) = 0, \quad u(l,t) = 0,$$

$$u(x,0) = u^0(x), \quad 0 < x < l.$$

Domain decomposition:

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega = \{x \mid 0 < x < l\}.$$



$$\Omega_1 = (0, b), \quad \Omega_2 = (a, l).$$

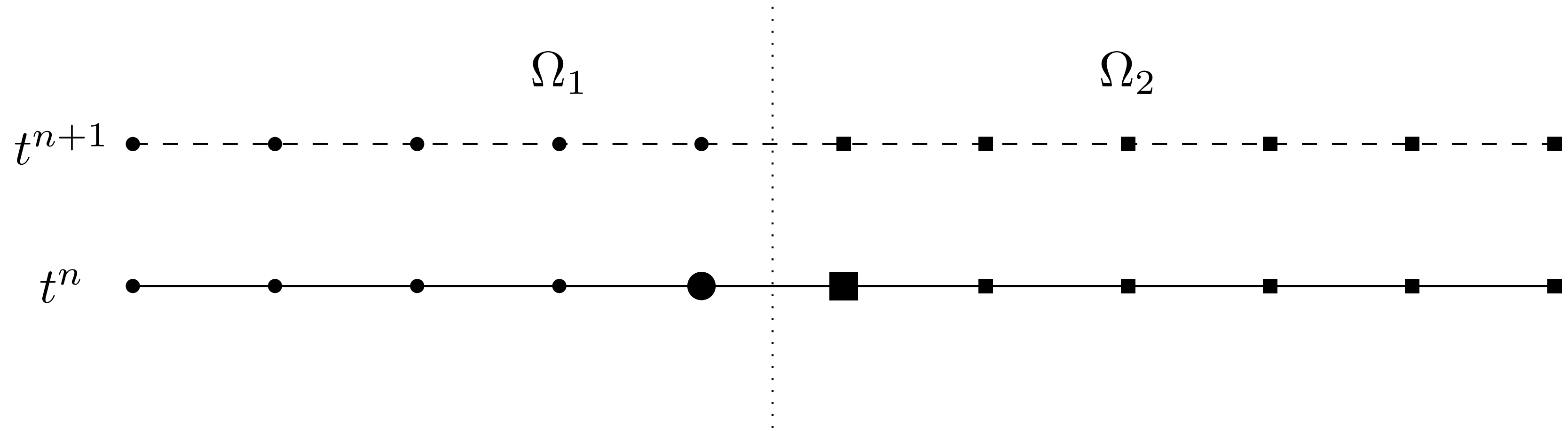
The minimal overlay (h — discretization steps in space):

$$b - a = h.$$

Decomposition with minimal overlap

Stencil:

The transition to the new time level:



Let $i = k$ и $i = k+1$ — nodes on the boundary of the subdomains:

$$\frac{y_i^{n+1} - y_i^n}{\tau} - \frac{y_{i+1}^{n+1} - 2y_i^{n+1} + y_{i-1}^{n+1}}{h^2} = 0, \quad i < k, \quad i > k+1,$$

$$\frac{y_k^{n+1} - y_k^n}{\tau} - \frac{y_{k+1}^{n} - 2y_k^{n+1} + y_{k-1}^{n+1}}{h^2} = 0,$$

$$\frac{y_{k+1}^{n+1} - y_{k+1}^n}{\tau} - \frac{y_{k+2}^{n+1} - 2y_{k+1}^{n+1} + y_{k}^{n}}{h^2} = 0.$$

- Unconditional stability (+);
- Conditional convergence (–): $\mathcal{O}(\tau h^{-1/2})$.

Standard approximation

In a bounded domain Ω an unknown function $u(\boldsymbol{x}, t)$ satisfies the following equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\boldsymbol{x}) \frac{\partial u}{\partial x_\alpha} \right) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T,$$

in which $k(\boldsymbol{x}) \ge \kappa > 0, \ \boldsymbol{x} \in \Omega$.
Homogeneous Dirichlet boundary conditions:

$$u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \le T.$$

The initial condition:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

Notations

Let $(\cdot,\cdot), \|\cdot\|$ be the scalar product and the norm in $L_2(\Omega)$:

$$(u,v) = \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})d\boldsymbol{x}, \quad \|u\| = (u,u)^{1/2}.$$

A symmetric positive definite bilinear form $d(u,v)$ such that

$$d(u,v) = d(v,u), \quad d(u,u) \geq \delta \|u\|^2, \quad \delta > 0,$$

is associated with the Hilbert space H_d equipped with the following scalar product and norm:

$$(u,v)_d = d(u,v), \quad \|u\|_d = (d(u,u))^{1/2}.$$

Suppose $t = t^n = n\tau,\ n = 0,1,...,$ where $\tau > 0$ is a constant time step.
A finite-dimensional space of finite elements is denoted by $\mathcal{V}^h$, and u^n $(u^n \in \mathcal{V}^h)$ stands for the approximate solution at the time level $t = t^n$.

The variational problem

$$\left(\frac{du}{dt}, v\right) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad 0 < t \leq T,$$

$$(u(0), v) = (u^0, v), \quad \forall v \in H_0^1(\Omega),$$

where

$$a(u, v) = \int_\Omega k(\boldsymbol{x}) \operatorname{grad} u \operatorname{grad} v \ d\boldsymbol{x}.$$

We study the projection-difference scheme

$$\left(\frac{y^{n+1}-y^n}{\tau}, v\right) + a(\sigma y^{n+1} + (1-\sigma) y^n, v) = (f(\sigma t^{n+1} + (1-\sigma) t^n), v),$$

$$\forall v \in \mathcal{V}^h, \quad n = 1, 2, ...,$$

where σ is a number (weight). If $\sigma = 0$, then the scheme is an explicit (forward-time) scheme, for $\sigma = 1$, we obtain a fully implicit (backward-time) scheme, and $\sigma = 0.5$ yields a symmetric (the so-called Crank–Nicolson) scheme. The condition

$$(v, v) + \left(\sigma - \frac{1}{2}\right) \tau \, a(v, v) \geq 0, \quad \forall v \in \mathcal{V}^h$$

is necessary and sufficient for the stability of the scheme in the space H_a.

Decomposition operators

Partition of unity of Ω

Domain decomposition scheme we associate with the partition of unity of the computational domain Ω. Let the domain Ω consists of p separate subdomains

$$\Omega = \Omega_1 \cup \Omega_2 \cup ... \cup \Omega_p.$$

Individual subdomains can overlap.
With a separate subdomain $\Omega_\alpha, \ \alpha = 1, 2, ..., p$ we associate the function $\eta_\alpha(\boldsymbol{x}), \ \alpha = 1, 2, ..., p$ such that

$$\eta_\alpha(\boldsymbol{x}) = \begin{cases} > 0, & \boldsymbol{x} \in \Omega_\alpha, \\ 0, & \boldsymbol{x} \notin \Omega_\alpha, \end{cases} \qquad \alpha = 1, 2, ..., p,$$

where

$$\sum_{\alpha=1}^{p} \eta_\alpha(\boldsymbol{x}) = 1, \quad \boldsymbol{x} \in \Omega.$$

Subdomain splitting

In

$$\left(\frac{du}{dt}, v\right) + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \quad 0 < t \le T$$

we have

$$a(u,v) = \sum_{\alpha=1}^{p} a_\alpha(u,v), \quad (f,v) = \sum_{\alpha=1}^{p} (f_\alpha, v),$$

Here

$$(f_\alpha, v) = \int_\Omega \eta_\alpha(\boldsymbol{x}) f(\boldsymbol{x},t) v \; d\boldsymbol{x}$$

and (standard decomposition)

$$a_\alpha(u,v) = \int_\Omega \eta_\alpha(\boldsymbol{x}) k(\boldsymbol{x}) \operatorname{grad} u \operatorname{grad} v \; d\boldsymbol{x}, \quad \alpha = 1, 2, ..., p.$$

Other variants

$$a_\alpha(u, v) = \int_\Omega k(\boldsymbol{x}) \operatorname{grad} u \operatorname{grad}(\eta_\alpha(\boldsymbol{x}) v) \ d\boldsymbol{x},$$

$$a_\alpha(u, v) = \int_\Omega k(\boldsymbol{x}) \operatorname{grad}(\eta_\alpha(\boldsymbol{x}) u) \operatorname{grad} v \ d\boldsymbol{x}, \quad \alpha = 1, 2, ..., p.$$

Operator form

We have

$$B\frac{dy}{dt} + Ay = \varphi(t), \quad 0 < t \leq T,$$

$$y(0) = y^0.$$

Here the mass matrix

$$B = B^* > 0,$$

the stiffness matrix

$$A = A^* > 0.$$

Operator splitting

We have

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad \varphi = \sum_{\alpha=1}^{p} \varphi_\alpha$$

with (standard decomposition)

$$A_\alpha = A_\alpha^* \geq 0, \quad \alpha = 1, 2, ..., p.$$

Symmetrized equation:

$$\frac{dw}{dt} + \widetilde{A} w = \widetilde{\varphi}(t), \quad 0 < t \leq T,$$

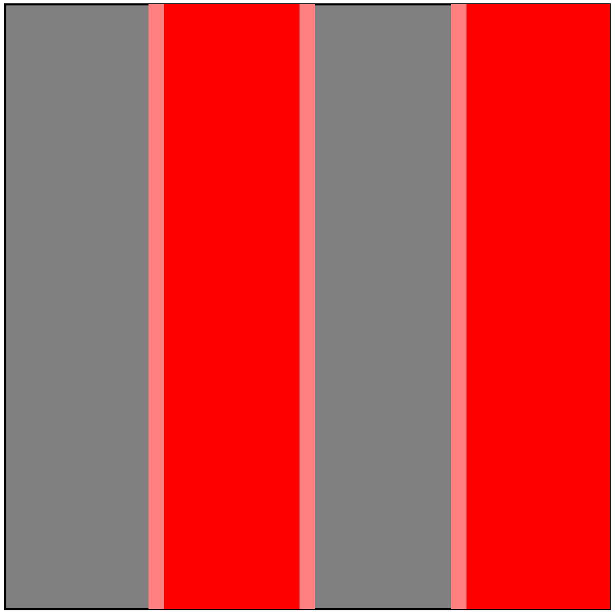
where

$$w = B^{1/2} v, \quad \widetilde{A} = B^{-1/2} A B^{-1/2}, \quad \widetilde{\varphi} = B^{-1/2} \varphi,$$

$$\widetilde{A} = \sum_{\alpha=1}^{p} \widetilde{A}_\alpha, \quad \widetilde{A}_\alpha = \widetilde{A}_\alpha^* = B^{-1/2} A_\alpha B^{-1/2}, \quad \alpha = 1, 2, ..., p.$$

Splitting schemes

Two-component splitting ($p = 2$)



- Douglas-Rachford scheme
- Peaceman–Rachford scheme
- Factorized scheme
- Symmetrical scheme of componentwise splitting

$$\left(\frac{u^{n+1/2} - u^n}{\tau}, v\right) + a_1(u^{n+1/2}, v) + a_2(u^n, v) = (f^{n+1}, v),$$

$$\left(\frac{u^{n+1} - u^n}{\tau}, v\right) + a_1(u^{n+1/2}, v) + a_2(u^{n+1}, v) = (f^{n+1}, v),$$

$$\forall v \in \mathcal{V}^h, \quad n = 1, 2, \ldots.$$

The problem in the subdomain (explicit-implicit scheme):

$$(u^{n+1/2}, v) + \tau a_1(u^{n+1/2}, v) = (\chi^n, v),$$

$$(u^{n+1}, v) + \tau a_2(u^{n+1}, v) = (\chi^{n+1/2}, v).$$

$$\left(\frac{u^{n+1/2} - u^n}{\tau}, v \right) + a_1(\sigma u^{n+1/2} + (1-\sigma)u^n, v) + a_2(u^n, v) = (f^{n+1/2}, v),$$

$$\left(\frac{u^{n+1} - u^n}{\tau}, v \right) + a_1(\sigma u^{n+1/2} + (1-\sigma)u^n, v) + a_2(\sigma u^{n+1} + (1-\sigma)u^n, v) = (f^{n+1/2}, v),$$

$$\forall v \in \mathcal{V}^h, \quad n = 1, 2, ...$$

$\sigma = 1/2$ — Peaceman-Rachford scheme,
$\sigma = 1$ — Douglas-Rachford scheme.

Factorized scheme

The operator form:

$$(B + \sigma\tau A_1) B^{-1} (B + \sigma\tau A_2) \frac{y^{n+1} - y^n}{\tau} + A y^n = \varphi^n.$$

Estimation of stability:

$$\|(B + \sigma\tau A_2) y^{n+1}\|_{B^{-1}} \leq \|(B + \sigma\tau A_2) y^n\|_{B^{-1}} + \tau \|\varphi^n\| \|_{B^{-1}}.$$

Theorem. For error factored regionally-additive difference scheme with $\sigma \geq 1/2$ estimate

$$\|(B + \sigma\tau A_2) z^{n+1}\|_{B^{-1}} \leq M \left(h^2 + \tau^2 + \left(\sigma - \frac{1}{2}\right)\tau + \sigma\tau \|\chi_2\|_A \right)$$

holds.
With minimal overlapping of subdomains this estimate gives

$$\|(B + \sigma\tau A_2) z^{n+1}\|_{B^{-1}} \leq M \left(h^2 + \tau^2 + \left(\sigma - \frac{1}{2}\right)\tau + \sigma\tau H^{-1/2} h^{-1/2} \right),$$

where H — step coarse mesh.
The scheme with $\sigma = 1/2$ does not increase order accuracy. Yet in this case the main error term is two times lower compared to $\sigma = 1$.

Multicomponent splitting

The main classes of additive schemes[4,5]:

- Schemes of componentwise splitting;
- Additevely averaged schemes of summarized approximation;
- Regularized additive schemes;
- Vector additive schemes.

[4] G. I. Marchuk, Splitting and alternating direction methods. Handbook of Numerical Analysis, Vol. I, pp. 197–462. North-Holland, 1990.

[5] P. N. Vabishchevich, Additive Additive Operator-Difference Schemes. Splitting Schemes. De Gruyter, 2013.

Scheme of componentwise splitting

The classic serial version:

$$\left(\frac{u^{n+\alpha/p} - u^{n+(\alpha-1)/p}}{\tau}, v\right) + a_\alpha(\sigma u^{n+\alpha/p} + (1-\sigma)u^{n+(\alpha-1)/p}, v) = (f_\alpha^{n+1}, v),$$

$$\alpha = 1, 2, ..., p.$$

The right side:

$$f_\alpha^{n+1} = \begin{cases} 0, & \alpha = 1, 2, ..., p-1, \\ f^{n+1}, & \alpha = p. \end{cases}$$

Unconditional stability when $\sigma \geq 0.5$.

Additevely averaged schemes:

$$\left(\frac{u_\alpha^{n+1} - u_\alpha^n}{p\tau}, v \right) + a_\alpha(\sigma u_\alpha^{n+1} + (1-\sigma) u_\alpha^n, v) = (f_\alpha^{n+1}, v),$$

$$\alpha = 1, 2, ..., p,$$

$$(u^{n+1}, v) = \frac{1}{p} \sum_{\alpha=1}^{p} (u_\alpha^{n+1}, v).$$

Vector problem

Instead of a single unknown $u(t)$ we consider p unknowns $u_\alpha,\ \alpha = 1, 2, ..., p$, which are determined from the system

$$\left(\frac{du_\alpha}{dt}, v \right) + \sum_{\beta=1}^{p} a_\beta(u_\beta, v) = (f, v), \quad \alpha = 1, 2, ..., p, \quad 0 < t \leq T.$$

For this system of equations are used the initial conditions

$$(u_\alpha(0), v) = (u^0, v), \quad \alpha = 1, 2, ..., p.$$

Each component is a solution to the original problem.

Unconditionally stable two-level scheme (the serial version):

$$\left(\frac{u_\alpha^{n+1} - u_\alpha^n}{\tau}, v \right) + \sum_{\beta=1}^{\alpha} a_\beta(u_\beta^{n+1}, v) + \sum_{\beta=\alpha+1}^{p} a_\beta(u_\beta^n, v) = (f^{n+1}, v),$$

$$(u_\alpha^0, v) = (u^0, v), \quad \alpha = 1, 2, ..., p.$$

Parallel version:

$$\left(\frac{u_\alpha^{n+1} - u_\alpha^n}{\tau}, v \right) + a_\alpha(\sigma u_\alpha^{n+1} + (1-\sigma) u_\alpha^n, v) + \sum_{\alpha \neq \beta = 1}^{p} a_\beta(u_\beta^n, v) = (f^{n+1}, v),$$

$$\alpha = 1, 2, ..., p.$$

Other problems

$$\frac{\partial^2 u}{\partial t^2} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha}\right) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T.$$

Boundary conditions:

$$u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \le T.$$

The initial condition:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \frac{\partial u}{\partial t}(\boldsymbol{x}, 0) = v^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

The symmetrical form:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{\alpha=1}^{m} \left(c_\alpha(\boldsymbol{x}, t) \frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} (c_\alpha(\boldsymbol{x}, t) u) \right)$$

$$- \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\boldsymbol{x}) \frac{\partial u}{\partial x_\alpha} \right) = f(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T.$$

Boundary conditions:

$$u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad 0 < t \le T.$$

The initial condition:

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega.$$

We write the unsteady convection-diffusion problems in the form

$$\frac{du}{dt} + \mathcal{A}u = f(t), \quad 0 < t \le T. \quad 0 < t \le T,$$

$$u(0) = u^0.$$

The operators of diffusive and convective transport are selected:

$$\mathcal{A} = \mathcal{C} + \mathcal{D},$$

$$\mathcal{D}u = -\sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\boldsymbol{x}) \frac{\partial u}{\partial x_\alpha} \right),$$

$$\mathcal{C}u = \frac{1}{2} \sum_{\alpha=1}^{m} \left(v_\alpha(\boldsymbol{x}, t) \frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} (v_\alpha(\boldsymbol{x}, t) u) \right).$$

Main properties:

$$\mathcal{D} = \mathcal{D}^* > 0, \quad \mathcal{C} = -\mathcal{C}^*.$$

DDM splitting

Additive representation of the operator $\mathcal{A}$:

$$\mathcal{A} = \sum_{\alpha=1}^{p} \mathcal{A}_\alpha, \quad \mathcal{A}_\alpha = \mathcal{C}_\alpha + \mathcal{D}_\alpha, \quad \alpha = 1, 2, ..., p,$$

in which

$$\mathcal{D}_\alpha u = - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\boldsymbol{x}) \eta_\alpha(\boldsymbol{x}) \frac{\partial u}{\partial x_\alpha} \right),$$

$$\mathcal{C}_\alpha u = \frac{1}{2} \sum_{\alpha=1}^{m} \left(v_\alpha(\boldsymbol{x}) \eta_\alpha(\boldsymbol{x}) \frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} (v_\alpha(\boldsymbol{x}) \eta_\alpha(\boldsymbol{x}) u) \right).$$

Self-adjoint part of operator $\mathcal{A}$ splits into the sum of nonnegative selfadjoint operators:

$$\mathcal{D}_\alpha = \mathcal{D}_\alpha^* \geq 0, \quad \alpha = 1, 2, ..., p.$$

Skew of the splitting in the sum of skew-symmetric operators:

$$\mathcal{C}_\alpha = -\mathcal{C}_\alpha^*, \quad \alpha = 1, 2, ..., p.$$

For the individual terms of operator:

$$\mathcal{A}_\alpha = \mathcal{C}_\alpha + \mathcal{D}_\alpha \geq 0, \quad \alpha = 1, 2, ..., p.$$