Isogeometric BDDC preconditioners with deluxe scaling

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Outline

- Motivations
- IGA discrete space: B-splines and NURBS
- BDDC method
- Deluxe scaling
- Numerical experiments

Motivations

- In engineering computing practice (automotive, aerospace and shipbuilding), bodies/domains are typically described with CAD (Computer Aided Design) using NURBS (Non Uniform Rational B-Splines) functions.
- In contrast, Finite Element Analysis (FEA) is based on piecewise polynomial functions $\longrightarrow$ mismatch between CAD and FEA different geometries.
- Possible solution: Isogeometric Analysis (IGA) uses CAD geometry and NURBS discrete spaces ($\sim$ *hpk*-fem).
- (CAD industry is about five times the FEM industry: quite unreasonable to expect a change in CAD industry).

Introduction

IGA very active emerging field, growing literature, see e.g.

J. A. Cottrell, T. J. R. Hughes, Y. Bazilevs, *Isogeometric Analysis. Toward integration of CAD and FEA*, Wiley, 2009 and subsequent works

Some recent references for IGA solvers:

- N. Collier, D. Pardo, L. Dalcin, M. Paszynski and V.M. Calo. *The cost of continuity: a study of the performance of isogeometric finite elements using direct solvers. Comput. Meth. Appl. Mech. Engrg.*, 2012
- N. Collier, D. Pardo, L. Dalcin, M. Paszynski and V.M. Calo. *The cost of continuity: performance of iterative solvers on isogeometric finite elements. SIAM J. Sci. Comp.*, 2013
- L. Beirão da Veiga, D. Cho, L. F. Pavarino, S. Scacchi, *Overlapping Schwarz methods for Isogeometric Analysis SIAM J. Numer. Anal.*, 2012
- L. Beirão da Veiga, D. Cho, L.F. Pavarino, S. Scacchi, *BDDC preconditioners for Isogeometric Analysis. M3AS*, 2013
- S. Kleiss, C. Pechstein, B. Juttler, S. Tomar, *IETI - Isogeometric Tearing and Interconnecting. RICAM TR*, 2012-01
- K. Gahalaut, J. Kraus, S. Tomar, *Multigrid methods for Isogeometric discretization. RICAM TR*, 2012-08

Notations for B-splines (2D)

- $\widehat{\Omega} = (0,1) \times (0,1)$ 2D parametric space.
- Knot vectors
  $\{\xi_1 = 0, \ldots, \xi_{n+p+1} = 1\}, \; \{\eta_1 = 0, \ldots, \eta_{m+q+1} = 1\}$,
  generate a mesh of rectangular elements in parametric space
- 1D basis functions N_i^p, M_j^q, $i = 1, ..., n$, $j = 1, ..., m$ of degree p and q, respectively, are defined from the knot vectors
- Bivariate spline basis on $\widehat{\Omega}$ is then defined by the tensor product

$$B_{i,j}^{p,q}(\xi, \eta) = N_i^p(\xi) \, M_j^q(\eta)$$

- 2D B-spline space:

$$\widehat{\mathcal{S}}_h = \text{span}\{B_{i,j}^{p,q}(\xi, \eta), \; i = 1, \ldots, n, \, j = 1, \ldots, m\}$$

- Analogously in 3D

Notations for NURBS

- 1D NURBS basis functions of degree p are defined by

$$R_i^p(\xi) = \frac{N_i^p(\xi)\omega_i}{w(\xi)},$$

where $w(\xi) = \sum_{\hat{i}=1}^{n} N_{\hat{i}}^p(\xi)\omega_{\hat{i}} \in \widehat{\mathcal{S}}_h$ is a fixed weight function

- 2D NURBS basis functions in parametric space $\widehat{\Omega} = (0,1)^2$

$$R_{i,j}^{p,q}(\xi,\eta) = \frac{B_{i,j}^{p,q}(\xi,\eta)\omega_{i,j}}{w(\xi,\eta)},$$

with $w(\xi,\eta) = \sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m} B_{\hat{i},\hat{j}}^{p,q}(\xi,\eta)\omega_{\hat{i},\hat{j}}$ fixed weight function,
$\omega_{i,j} = (\mathbf{C}_{i,j}^{\omega})_3$ and $\mathbf{C}_{i,j}$ a mesh of $n \times m$ control points

Define the geometrical map $\mathbf{F} : \widehat{\Omega} \to \Omega$ given by

$$\mathbf{F}(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}^{p,q}(\xi, \eta) \mathbf{C}_{i,j}.$$

Space of NURBS scalar fields on a single-patch domain Ω (NURB region) is the span of the *push-forward* of 2D NURBS basis functions (as in isoparametric approach)

$$\mathcal{N}_h = \operatorname{span}\{R_{i,j}^{p,q} \circ \mathbf{F}^{-1}, i = 1, \ldots, n, j = 1, \ldots, m\}.$$

The image of the elements in the parametric space are elements in the physical space. The physical mesh on Ω is therefore

$$\mathcal{T}_h = \{\mathbf{F}((\xi_i, \xi_{i+1}) \times (\eta_j, \eta_{j+1})),\ i = 1, \ldots, n+p, j = 1, \ldots, m+q\},$$

where the empty elements are not considered.

Model problem

We will consider the following problem

$$\begin{cases} -\mathrm{div}(\rho \nabla u) & = & f & \text{in} & \Omega, \\ u & = & g & \text{su} & \partial\Omega_D \\ \nabla u \cdot \mathbf{n} & = & 0 & \text{su} & \partial\Omega_N \end{cases}$$

with Ω a bounded and connected CAD domain $\subset \mathbb{R}^d$, $d = 2, 3$ and ρ a scalar field satisfying $0 < \rho_m \leq \rho(x) \leq \rho_M, \ \forall x \in \Omega$.

Variational formulation in the NURBS discrete space $\mathcal{N}_h$ living in the physical space.

Resulting discrete problem $Au = b$ is solved iteratively by PCG.

BDDC preconditioner built by decomposing the spline functions in parametric space.

Domain decomposition in parametric space

- Select a subset $\{\xi_{i_k}, k = 1, \ldots, N+1\}$, $\xi_{i_k} \neq \xi_{i_{k+1}}$, with $\xi_{i_1} = 0, \xi_{i_{N+1}} = 1$ from the full set of knots.
- ξ_{i_k}, $k = 2, .., N$ are the interface knots

$$\overline{(\widehat{I})} = [0,1] = \overline{\Big(\bigcup_{k=1,..,N} \widehat{I}_k\Big)}, \quad \text{with} \quad \widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}}),$$

- $H_k = \mathrm{diam}(\widehat{I}_k)$, $H = \max_k H_k$.
- In more dimensions, just use the tensor product

$$\begin{array}{l} \widehat{I}_k = (\xi_{i_k}, \xi_{i_{k+1}}), \quad \widehat{I}_l = (\eta_{j_l}, \eta_{j_{l+1}}), \\ \widehat{\Omega}_{kl} = \widehat{I}_k \times \widehat{I}_l, \quad 1 \leq k \leq N_1,\ 1 \leq l \leq N_2. \end{array}$$

- Parametric space is then decomposed as

$$\overline{\widehat{\Omega}} = \bigcup_{k=1,..,N_1 N_2} \overline{\widehat{\Omega}}_k,$$

Splitting of degrees of freedom

- As in classical iterative substructuring, we can define the interface

$$\Gamma = \bigcup_{j \neq k} \partial\widehat{\Omega}_k \cap \partial\widehat{\Omega}_j$$

- Internal (I) and interface (Γ) degrees of freedom

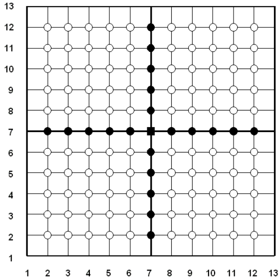
$$\begin{aligned}
\Theta_\Omega &= \{(i,j) \in \mathbb{N}^2 : \ 1 \leq i \leq n, \ 1 \leq j \leq m\}, \\
\Theta_\Gamma &= \{(i,j) \in \Theta_\Omega : \ \mathrm{supp}(B_{i,j}^{p,q}) \cap \Gamma \neq \emptyset\}, \\
\Theta_I &= \Theta_\Omega \setminus \Theta_\Gamma,
\end{aligned}$$

- ... and their local counterparts

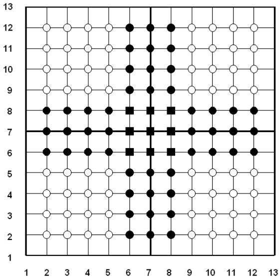
$$\begin{aligned}
\Theta_\Omega^{(kl)} &= \{(i,j) \in \mathbb{N}^2 : \ i_k \leq i \leq i_{k+1}, \ j_l \leq j \leq j_{l+i}\}, \\
\Theta_\Gamma^{(kl)} &= \{(i,j) \in \Theta_\Omega^{(kl)} : \ \mathrm{supp}(B_{i,j}^{p,q}) \cap \Gamma \neq \emptyset\}, \\
\Theta_I^{(kl)} &= \Theta_\Omega^{(kl)} \setminus \Theta_\Gamma^{(kl)},
\end{aligned}$$

"Fat" boundaries

The high continuity of IGA functions requires us to introduce the concept of a fat interface: 2×2 example



C^0 splines



C^2 splines

$\circ$ = interior index set

$\bullet$ = interface index set

$\blacksquare$ = vertex (primal) index set

Discrete function spaces

- Local function spaces can then be defined

$$\begin{aligned} \mathbf{W}_I^{(k)} &= \text{span}\{R_{i,j}^{p,q} \ : \ (i,j) \in \Theta_I^{(k)}\}, \\ \mathbf{W}_\Gamma^{(k)} &= \text{span}\{R_{i,j}^{p,q} \ : \ (i,j) \in \Theta_\Gamma^{(k)}\}, \\ \mathbf{W}_I^{(k)} &= \text{span}\{R_{i,j}^{p,q} \ : \ (i,j) \in \Theta_I^{(k)}\} \end{aligned}$$

- ... together with the usual product spaces

$$\mathbf{W}_I = \prod_{k=1}^{K} \mathbf{W}_I^{(k)}, \quad \mathbf{W}_\Gamma = \prod_{k=1}^{K} \mathbf{W}_\Gamma^{(k)}.$$

with $\widehat{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$ the subset of continuos functions.

- local Neumann stiffness matrices are then assembled

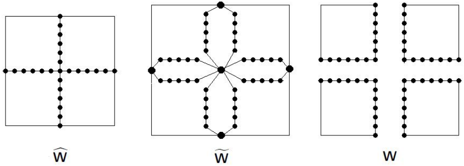
$$A^{(k)} = \begin{bmatrix} A_{II}^{(k)} & A_{I\Gamma}^{(k)} \\ A_{\Gamma I}^{(k)} & A_{\Gamma\Gamma}^{(k)} \end{bmatrix}$$

BDDC (Balancing Domain Decomposition by Constraints)

- BDDC is an evolution of Balancing Neumann-Neumann
  - C. Dohrmann SISC 25, 2003
  - J. Mandel, C. Dohrmann, NLAA 10, 2003
  - J. Mandel, C. Dohrmann, R. Tezaur, Appl. Numer. Math. 54, 2005
- BDDC method algebraically defined by selecting from $\widehat{\mathbf{W}}_\Gamma$ a set of primal dofs plus a proper scaling.
- Proper choice of primal dofs guaranties well-posedness, scalablity and quasi-optimality of the method
- Dual of FETI-DP preconditioners with same primal set, since both have essentially the same spectrum.
  - C. Farhat, M. Lesoinne, P. Le Tallec, K. Pierson, D. Rixen, IJNME 50, 2001
- BDDC for IGA
  - Beirao da Veiga, Cho, Pavarino, Scacchi, M3AS 23, 2013.

- Define the space $\widetilde{\mathbf{W}}_\Gamma$, intermediate between $\widehat{\mathbf{W}}_\Gamma$ and $\mathbf{W}_\Gamma$, by the splitting of $\mathbf{W}_\Gamma$ into primal (Π) and dual (Δ) spaces

$$\widetilde{\mathbf{W}}_\Gamma = \mathbf{W}_\Delta \oplus \widehat{\mathbf{W}}_\Pi, \quad \widetilde{\mathbf{W}} = \mathbf{W}_I \oplus \widetilde{\mathbf{W}}_\Gamma, \quad \widehat{\mathbf{W}} \subset \widetilde{\mathbf{W}} \subset \mathbf{W}.$$



- $\widehat{\mathbf{W}}_\Pi$ subspace of functions continuous at primal dofs
- $\mathbf{W}_\Delta = \prod \mathbf{W}_\Delta^{(i)}$ the product space of interface functions vanishing at primal dofs.

BDDC method: partial subassembling

BDDC considers the partially subassembled matrix on $\widetilde{\mathbf{W}}$

$$\widetilde{A} = \begin{bmatrix} A_{II}^{(1)} & A_{I\Delta}^{(1)} & & & & \widetilde{A}_{I\Pi}^{(1)} \\ A_{I\Delta}^{(1)\,T} & A_{\Delta\Delta}^{(1)} & & & & \widetilde{A}_{\Delta\Pi}^{(1)} \\ & & \ddots & & & \vdots \\ & & & A_{II}^{(N)} & A_{I\Delta}^{(N)} & \widetilde{A}_{I\Pi}^{(N)} \\ & & & A_{I\Delta}^{(N)\,T} & A_{\Delta\Delta}^{(N)} & \widetilde{A}_{\Delta\Pi}^{(N)} \\ \widetilde{A}_{I\Pi}^{(1)\,T} & \widetilde{A}_{\Delta\Pi}^{(1)\,T} & \dots & \widetilde{A}_{I\Pi}^{(N)\,T} & \widetilde{A}_{\Delta\Pi}^{(N)\,T} & \widetilde{A}_{\Pi\Pi} \end{bmatrix} = \begin{bmatrix} A_{rr} & \widetilde{A}_{r\Pi} \\ \widetilde{A}_{r\Pi}^{T} & \widetilde{A}_{\Pi\Pi} \end{bmatrix}$$

where

$$\widetilde{A}_{I\Pi}^{(j)} = A_{I\Pi}^{(j)} R_{\Pi}^{(j)}, \quad \widetilde{A}_{\Delta\Pi}^{(j)} = A_{\Delta\Pi}^{(j)} R_{\Pi}^{(j)}$$

$$\widetilde{A}_{\Pi\Pi} = \sum_{j=1}^{N} R_{\Pi}^{(j)\,T} A_{\Pi\Pi}^{(j)} R_{\Pi}^{(j)}.$$

BDDC method: preconditioner application

BDDC preconditioner for A defined as

$$M_{BDDC}^{-1} = P_I + (I - P_I A)\widetilde{R}_D^T \widetilde{A}^{-1} \widetilde{R}_D (I - AP_I).$$

where by static condensation we obtain

$$\widetilde{A}^{-1} = \overline{R}_r^T A_{rr}^{-1} \overline{R}_r + \Phi S_{\Pi\Pi}^{-1} \Phi^T.$$

- Dirichlet solver: $P_I = R_I^T A_{II}^{-1} R_I$, $A_{II} = \mathrm{diag}(A_{II}^{(j)})$.
- Neumann solver: $\overline{R}_r^T A_{rr}^{-1} \overline{R}_r$.
- Coarse solver: $\Phi S_{\Pi\Pi}^{-1} \Phi^T$
  - Primal basis matrix: $\Phi = \overline{R}_\Pi^T - \overline{R}_r^T A_{rr}^{-1} \widetilde{A}_{r\Pi}$,
  - Coarse matrix: $S_{\Pi\Pi} = \Phi^T \widetilde{K} \Phi$.

[C. Dohrmann, NLAA 14, 2007]

[J. Li, O. B. Widlund, IJNME 196 2008]

BDDC method: scaling operator

- The scaled restriction operator

$$\widetilde{R}_D : \widehat{\mathbf{W}} \rightarrow \widetilde{\mathbf{W}}$$

restricts to the partially subassembled space and then multiplies dual dofs by a scaling matrix.

- Usual choice for $D^{(j)}$: diagonal scaling matrix with diagonal

$$\delta_j^{\dagger}(x) = \frac{\delta_j(x)}{\sum_{k \in \mathcal{N}_x} \delta_k(x)}.$$

- Possible choices of $\delta_j(x)$
  - ρ–scaling: $\delta_j(x) = 1$,
  - stiffness–scaling: $\delta_j(x) =$"diagonal entry of $A^{(j)}$ w.r.t. x".

BDDC method: deluxe scaling

- $D^{(k)}$ is a block diagonal matrix, with blocks given by

$$D_{\mathcal{F}}^{(k)} = \left(S_{\mathcal{F}}^{(k)} + S_{\mathcal{F}}^{(j)}\right)^{-1} S_{\mathcal{F}}^{(k)},$$

  with $\mathcal{F}$ a 2D edge or 3D face shared by the subdomains Ω_k and Ω_j (similarly for edges in 3D).

- $S_{\mathcal{F}}^{(k)}$ and $S_{\mathcal{F}}^{(j)}$ are obtained from $S^{(k)}$ and $S^{(j)}$ by "extracting" all rows and columns which belong to $\mathcal{F}$.
- The action of $\left(S_{\mathcal{F}}^{(k)} + S_{\mathcal{F}}^{(j)}\right)^{-1}$ can be computed by solving a Dirichlet problem on $\Omega_k \cup \mathcal{F} \cup \Omega_j$ (with nonzero r.h.s. on $\mathcal{F}$ dofs only)

[Dohrmann, Widlund. DD20 2011; Dohrmann DD21, 2012; Oh, Wildlund, Dohrmann, TR2013-951, Courant Institte NY, 2013 Oh and Kym talks at MS-1. Olof's plenary on thursday morning]

Deluxe scaling: parallel implementation

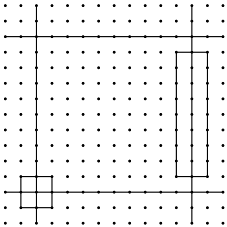
- Parallel PETSc code assumes one subdomain per MPI process
- Deluxe subproblems:
  - faces: $\left(S_{\mathcal{F}}^{(k)} + S_{\mathcal{F}}^{(j)}\right)^{-1} S_{\mathcal{F}}^{(k)}$,
  - edges: $\left(\sum_{j \in \mathcal{N}_{\mathcal{E}}} S_{\mathcal{E}}^{(j)}\right)^{-1} S_{\mathcal{E}}^{(k)}$.
- Smaller subproblems computed explicitly
- Larger subproblems solved in parallel with MUMPS on subcommunicators
- Parallelization achieved by distance-1 coloring of connectivity graph of the interface equivalence classes
  - Each vertex is a deluxe subproblem (w.r.t. $\mathcal{F}$ or $\mathcal{E}$)
  - 2 vertices are connected iff they share the same subdomain
  - Graph coloring using COLPACK (C++, STL).
- Bounded number of colors with increasing number of subdomains

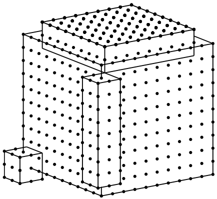
K	2^3	3^3	4^3	5^3	6^3	7^3	8^3
problems	18	90	252	540	990	1638	2520
colors	6	18	18	19	18	20	19

Choice of coarse space

We will consider the following spaces

- $\widehat{V}_{\Pi}^{C}$: containing all fat dofs belonging to fat vertices
- $\widehat{V}_{\Pi}^{C+E}$: $\widehat{V}_{\Pi}^{C}$ augmented with "slim" edge averages
- $\widehat{V}_{\Pi}^{C+E+F}$: $\widehat{V}_{\Pi}^{C+E}$ augmented with "slim" face averages





IGA BDDC convergence rate bounds (2D only)

Theorem

Using $\widehat{V}_{\Pi}^{C}$ the condition number of the BDDC preconditioned isogeometric operator is bounded by

$$
\begin{aligned}
\kappa_2(M^{-1}S_\Gamma) \leq &\quad C_\rho \left(1+\log^2\left(\tfrac{H}{h}\right)\right) && \textit{for } \rho\textit{-scaling} \\
\kappa_2(M^{-1}S_\Gamma) \leq &\quad C_s \left(1+\log\left(\tfrac{H}{h}\right)\right) \tfrac{H}{h} && \textit{for stiffness scaling} \\
\kappa_2(M^{-1}S_\Gamma) \leq &\quad C_d \left(1+\log^2\left(\tfrac{H}{h}\right)\right) && \textit{for deluxe scaling}
\end{aligned}
$$

with $C_\bullet$ constants independent on h, H, K (but not on p and k).

More details and proofs in:

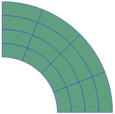
Beirao da Veiga, Cho, Pavarino, Scacchi, M3AS 23, 2013.

Beirao da Veiga, Pavarino, Scacchi, Widlund, Zampini, *Submitted*.

Experimental setting (with advertisements)

- Dirichlet boundary condition on one side, Neuman on the rest
- PCG with null initial guess, random rhs, *rtol* =1E-6
- 2D results obtained in Matlab using GeoPDEs
  [De Falco, Reali, Vazquez. TR 22PV10/20/0 IMATI-CNR, 2010]
- 3D Results obtained with PETSc code on IBM BlueGene/Q FERMI at CINECA (http://www.hpc.cineca.it/)
  - 10.240 PowerA2 sockets @1.6GHz
  - 163.840 compute cores, Rpeak 2.1 PFlops (12th in top500 06/13)
  - Current prace project call (8th) open until 15th October 2013
    Take a look at http://www.prace-project.eu/

BDDC results, 2D quarter ring domain



BDDC prec. $p = 3, k = 2$ NURBS, quarter ring domain									
		$1/h = 16$		$1/h = 32$		$1/h = 64$		$1/h = 128$	
	K	κ_2	it.	κ_2	it.	κ_2	it.	κ_2	it.
ρ	2×2	74.94	34	75.85	43	76.17	46	76.30	51
	4×4			78.88	51	76.52	55	76.30	58
	8×8					78.34	58	76.39	59
	16×16							78.47	59
stiffness	2×2	3.67	12	5.53	14	11.29	17	23.23	23
	4×4			5.62	16	14.33	20	35.49	29
	8×8					6.57	18	16.87	29
	16×16							7.16	19
deluxe	2×2	1.24	5	1.42	6	1.65	6	1.92	6
	4×4			2.02	8	2.68	10	3.46	11
	8×8					2.39	10	3.29	12
	16×16							2.64	11

BDDC results, 2D square domain

BDDC prec. $p = 5, k = 4$ B-spline, square domain									
		$1/h = 16$		$1/h = 32$		$1/h = 64$		$1/h = 128$	
	K	κ_2	it.	κ_2	it.	κ_2	it.	κ_2	it.
ρ	2×2	5.3e4	95	5.4e4	124	5.4e4	166	5.4e4	181
	4×4			6.6e4	220	5.4e4	211	5.4e4	220
	8×8					6.8e4	291	5.4e4	235
	16×16							6.9e4	327
stiffness	2×2	17.43	23	17.41	24	17.41	23	18.35	25
	4×4			18.26	28	17.44	26	17.42	26
	8×8					18.51	28	17.44	26
	16×16							18.56	29
deluxe	2×2	1.19	5	1.35	6	1.55	6	1.78	6
	4×4			1.62	8	2.19	9	2.86	10
	8×8					1.77	8	2.55	10
	16×16							1.87	8

BDDC dependence on p, 2D domain

$k = p-1,\ k_\Gamma = 1,\ 1/h = 64,\ 1/H = 4$ B-splines, square domain								
p	Full		Schur		BDDC–ρ		BDDC–stiffness	
	κ_2	it.	κ_2	it.	κ_2	it.	κ_2	it.
2	311.46	72	72.57	37	3.40	12	7.61	16
3	366.81	77	75.99	41	3.82	12	6.12	14
4	477.38	114	90.27	51	4.25	12	5.68	13
5	1.78e3	275	114.56	73	4.61	12	5.79	13
6	1.49e4	774	338.43	122	4.92	12	6.05	13
7	1.27e5	2151	1.00e3	202	5.24	12	6.34	12
8	1.11e6	5907	2.98e3	337	5.50	12	6.64	12
9	1.01e7	16541	8.77e3	545	5.73	12	6.96	13
10	9.42e7	47050	2.56e4	868	5.95	12	7.24	13

$k = p-1,\ h = 1/64,\ 1/H = 4$ NURBS, quarter-ring domain, BDDC–deluxe									
p	2	3	4	5	6	7	8	9	10
κ_2	3.22	2.68	2.41	2.19	2.04	1.91	1.80	1.72	1.62
it .	10	10	9	9	9	8	8	8	9

BDDC dependence on p, 3D square domain

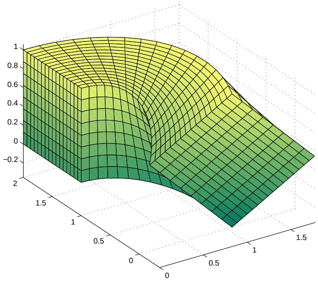
$k = p - 1,\ h = 1/24,\ H = 1/2$							
	p	2	3	4	5	6	7
		BDDC–deluxe					
$\widehat{V}_{\Pi}^{C}$	κ_2	5.62	4.71	4.39	3.92	5.12	11.15
	it.	12	11	12	14	18	26
$\widehat{V}_{\Pi}^{C+E}$	κ_2	2.10	1.91	2.03	2.68	4.99	10.92
	it.	10	9	10	12	17	26
$\widehat{V}_{\Pi}^{C+E+F}$	κ_2	1.58	1.45	1.70	2.68	4.99	10.92
	it.	8	8	9	12	17	26

BDDC quasi-optimality, 3D square domain

$p = 3, k = 2,\ K = 4 \times 4 \times 4$						
	H/h	4	8	12	16	20
		BDDC–stiffness				
$\widehat{V}_\Pi^C$	κ_2	7.03	10.59	21.30	34.64	46.97
	it.	24	26	34	40	46
$\widehat{V}_\Pi^{C+E}$	κ_2	6.35	6.09	6.13	6.26	8.15
	it.	23	22	22	23	26
$\widehat{V}_\Pi^{C+E+F}$	κ_2	6.35	6.09	6.13	6.16	6.19
	it.	23	22	22	21	22
		BDDC–deluxe				
$\widehat{V}_\Pi^C$	κ_2	2.62	6.13	10.10	14.19	17.91
	it.	12	18	21	24	27
$\widehat{V}_\Pi^{C+E}$	κ_2	1.54	1.80	2.03	2.21	2.35
	it.	9	10	11	12	12
$\widehat{V}_\Pi^{C+E+F}$	κ_2	1.54	1.37	1.46	1.62	1.75
	it.	9	8	8	9	9

	K	2^3	3^3	4^3	5^3	6^3	7^3	8^3	9^3	10^3
$p=3, k=2,\ H/h=8$										
BDDC–stiffness										
$\widehat{V}_\Pi^C$	κ_2	20.09	19.24	19.16	19.16	19.16	19.16	19.16	19.16	19.17
	it.	26	33	38	39	39	39	39	39	39
$\widehat{V}_\Pi^{C+E}$	κ_2	6.04	6.08	6.08	6.10	6.09	6.10	6.09	6.10	6.10
	it.	21	22	22	22	22	23	22	23	22
$\widehat{V}_\Pi^{C+E+F}$	κ_2	6.04	6.08	6.08	6.10	6.09	6.10	6.09	6.10	6.10
	it.	21	22	22	22	22	23	22	23	22
BDDC–deluxe										
$\widehat{V}_\Pi^C$	κ_2	8.96	8.38	8.44	8.38	8.35	8.35	8.35	8.36	8.35
	it.	20	21	23	24	23	23	24	24	24
$\widehat{V}_\Pi^{C+E}$	κ_2	2.06	2.01	1.98	1.98	1.98	1.98	1.98	1.98	1.98
	it.	10	11	11	10	10	10	10	10	10
$\widehat{V}_\Pi^{C+E+F}$	κ_2	1.42	1.40	1.41	1.40	1.40	1.40	1.40	1.40	1.40
	it.	8	8	8	8	8	8	8	8	8

BDDC scalability, 3D twisted-bar domain



$p = 3, k = 2,\ H/h = 6$						
	K	2^3	3^3	4^3	5^3	6^3
		BDDC–stiffness				
$\widehat{V}_\Pi^{C}$	κ_2	9.39	11.07	12.97	13.87	14.39
	it.	24	29	30	31	33
$\widehat{V}_\Pi^{C+E}$	κ_2	8.94	9.21	9.27	9.35	9.38
	it.	24	27	28	28	29
$\widehat{V}_\Pi^{C+E+F}$	κ_2	8.94	9.21	9.27	9.35	9.38
	it.	24	27	28	28	29
		BDDC–deluxe				
$\widehat{V}_\Pi^{C}$	κ_2	3.94	5.72	6.87	7.47	7.83
	it.	11	15	20	21	23
$\widehat{V}_\Pi^{C+E}$	κ_2	1.67	1.81	1.85	1.86	1.92
	it.	9	10	10	10	10
$\widehat{V}_\Pi^{C+E+F}$	κ_2	1.42	1.58	1.66	1.72	1.76
	it.	8	9	9	9	9

Conclusions

- BDDC preconditioners with deluxe scaling successfully extended to IGA
- Theory yields h-version convergence rate bounds for 2D case analogous to FEM case (p, k analysis still open), confirmed by numerical results
- Design of a minimal coarse space in 3D is still an open issue.
- Deluxe scaling becomes computationally effective with larger values of p and k in 3D.
- Extensions to almost incompressible elasticity and Stokes are in progress [Pavarino et al., MS1]
- BDDC and FETI-DP methods available within PETSc
- Support for BDDC solver available in PetIGA [N. Collier, L. Dalcin, V.M. Calo, 2013]