Sweeping Preconditioners and Source Transfer in the context of Domain Decomposition

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Optimal Schwarz Methods I

In their 1994 report

Optimal Interface Conditions for Domain Decomposition Methods,

Nataf–Rogier–Sturler formulated the Jacobi-type optimal Schwarz method

- for one-way (i.e. strip-wise or slice-wise) domain decomposition,
- that converges in J steps for J subdomains
- of which on interfaces the DtN transmission conditions are used.

Following this line, many works have focused on approximation of DtN, e.g. A. Toselli tried PML (though transmission conditions are put on most exterior boundary of PML so not a proper way from our point of view) in DD9, 1998; A. Schadle, L. Zschiedrich also tried PML in DD16, 2007 (which is the proper way from our viewpoint).

Optimal Schwarz Methods II

In their DD19, 2009 paper

Optimal Interface Conditions for an Arbitrary Decomposition into Subdomains,

Gander–Kwok formulated another Jacobi-type optimal Schwarz method

- that converges in 2 steps for J arbitrarily decomposed subdomains
- of which on interfaces all-to-all communications are used.

There has *not* been so far an approximation of the all-to-all operator.

AILU

In their 2001 paper,

AILU for Helmholtz Problems: a New Preconditioner Based on the Analytic Parabolic Factorization,

Gander–Nataf identified the parabolic factorization

$$\partial_{xx} + (\partial_{yy} + \omega^2) = (\partial_x + \mathrm{DtN}) \circ (\partial_x - \mathrm{DtN}), \quad \mathrm{DtN} = \mathbf{i}\sqrt{\partial_{yy} + \omega^2}$$

as the block LU factorization of the block tri-diagonal linear system.
A second-order local (i.e. differential) approximation of DtN was used.

A matrix equation for DtN (from the 2001 AILU paper)

The algebraic DtN (i.e. Schur complement) was found to satisfy a matrix equation. Consider elimination of a semi-infinite matrix from the beginning

$$\begin{pmatrix} \ddots & \ddots & & & \\ \ddots & D & L & & \\ & L & D & L & \\ & & L & D & A_{\Gamma I} \\ & & & A_{I\Gamma} & A_{II} \end{pmatrix} \begin{pmatrix} u_{\Gamma,1} \\ \vdots \\ \vdots \\ u_{\Gamma,\infty} \\ u_I \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{\Gamma,\infty} \\ f_I \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} T_\infty & A_{\Gamma I} \\ A_{I\Gamma} & A_{II} \end{pmatrix} \begin{pmatrix} u_{\Gamma,\infty} \\ u_I \end{pmatrix} = \begin{pmatrix} f_{\Gamma,\infty} \\ f_I \end{pmatrix}$$

$$T_N = D - L T_{N-1}^{-1} L, \quad T_1 = D.$$

Let $N \to \infty$, we obtain $T_\infty = D - L T_\infty^{-1} L$. This is less exploited than analytic techniques.

Sweeping Preconditioners of Engquist–Ying

In 2010, Engquist–Ying proposed their sweeping preconditioners.

- The connection between block LU and DtN was rediscovered.
- The improvement on AILU is achieved by using PML approximation of DtN rather than second-order differential.
- Artificial damping (introduced by S. Kim in 1996) was used for heterogeneous media.

Many works have been inspired. For examples,

- Poulson–Engquist–Fomel–Li–Ying's parallel implementation,
- Chen–Xiang's source transfer method with theory,
- Stolk's rapidly converging DDM,
- Childs–Graham–Shank's hybrid sweeping addressing parallelism.
- Geuzaine–Vion accelerate Jacobi-type OSM by GS-type

How can we understand all the variants in a unified way?

We will show that

- AILU,
- sweeping preconditioners,
- source transfer,
- and Stolk's DDM

are optimized Schwarz methods of symmetric Gauss-Seidel type with different choices of

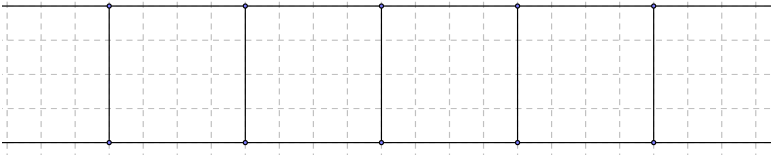
- transmission conditions on subdomain interfaces,
- and overlap size.

One-way Decomposition

Let us consider a PDE problem

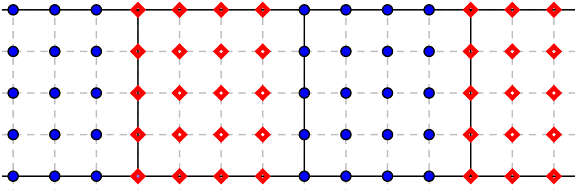
$$\begin{aligned} \mathcal{L}\, u &= f \quad \text{in } \Omega, \\ \mathcal{B}\, u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

We decompose Ω into strips (or slices in 3-D).



Block Tri-Diagonal System

Correspondingly, we introduce the partition of grids.



The linear system from discretization of the PDE then can be written in the block tri-diagonal form

$$\begin{pmatrix} A_{11} & A_{12} & & & \\ A_{21} & A_{22} & A_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & A_{J-1,J} \\ & & & A_{J,J-1} & A_{JJ} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{J-1} \\ u_J \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{J-1} \\ f_J \end{pmatrix},$$

Block LU Factorization

$$\begin{pmatrix} A_{11} & A_{12} & \\ A_{21} & A_{22} & A_{23} \\ & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} T_1 & & \\ A_{21} & T_2 & \\ & A_{32} & T_3 \end{pmatrix} \begin{pmatrix} I & T_1^{-1}A_{12} & \\ & I & T_2^{-1}A_{23} \\ & & I \end{pmatrix}$$

'L' Solve: the Forward Sweeping

$$\begin{pmatrix} T_1 & & & \\ A_{21} & T_2 & & \\ & A_{32} & T_3 & \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

Algorithm 1.1 Forward sweeping

Solve successively the following sub-problems:

$T_1 v_1 = f_1,$

$T_2 v_2 = f_2 - A_{21} v_1,$

$T_3 v_3 = f_3 - A_{32} v_2.$

'U' Solve: the Backward Sweeping

$$\begin{pmatrix} I & T_1^{-1}A_{12} & \\ & I & T_2^{-1}A_{23} \\ & & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Algorithm 1.2 Backward sweeping

Let $u_3 \leftarrow v_3$.
Solve successively

$$T_2 u_2 = T_2 v_2 - A_{23} u_3,$$
$$T_1 u_1 = T_1 v_1 - A_{12} u_2.$$

Block LU is an Optimal Schwarz Method

Theorem 1 (Gander, Z. '13)

The forward and backward sweeping based on the block LU factorization is a non-overlapping optimal Schwarz method of symmetric Gauss-Seidel type with DtN transmission conditions on the left boundaries of subdomains and Dirichlet transmission on the right and using zero initial guess.

Note that the optimal Schwarz method converges for one forward and one backward sweeping for arbitrary initial guess.

AILU, Sweeping Preconditioners are Optimized Schwarz Methods (OSM)

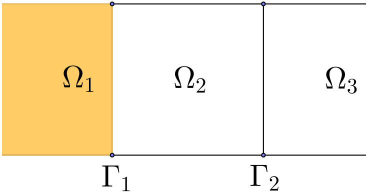
Corollary 2

AILU is a non-overlapping optimized Schwarz method of symmetric Gauss-Seidel type with second-order differential approximation of DtN transmission on the left boundaries of subdomains and Dirichlet on the right.

Corollary 3

Sweeping preconditioners of Engquist–Ying are non-overlapping optimized Schwarz methods of symmetric Gauss-Seidel type with PML or hierarchical matrix approximation of DtN transmission on the left boundaries of subdomains and Dirichlet on the right.

Schwarz Form of 'L' Solve at PDE Level

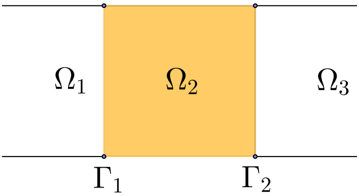


Algorithm 2.1 Schwarz form of the 'L' solve (taking $u^0 = 0$)

Given initial guess u^0, solve successively the following sub-problems:
(i)

$$\begin{cases} \mathcal{L}v_1 = f, & \text{in } \Omega_1, \\ \mathcal{B}v_1 = g, & \text{on } \partial\Omega, \\ v_1 = u_2^0, & \text{on } \Gamma_1, \end{cases}$$

Schwarz Form of 'L' Solve at PDE Level



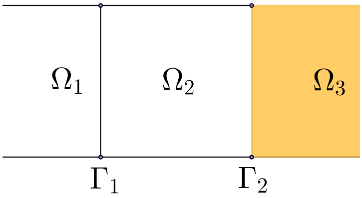
Algorithm 2.1 Schwarz form of the 'L' solve (taking $u^0 = 0$)

Given initial guess u^0, solve successively the following sub-problems:
(ii)

$$\begin{cases} \mathcal{L} v_2 = f, & \text{in } \Omega_2, \\ \mathcal{B} v_2 = g, & \text{on } \partial\Omega, \\ (\partial_{n_2} + \mathcal{S}_1^l)(v_2 - v_1) = 0, & \text{on } \Gamma_1, \\ v_2 = u_3^0, & \text{on } \Gamma_2, \end{cases}$$

where $\mathcal{S}_1^l$ is the DtN operator on Γ_1 calculated with $\mathcal{L}$ in Ω_1.

Schwarz Form of 'L' Solve at PDE Level



Algorithm 2.1 Schwarz form of the 'L' solve (taking $u^0 = 0$)

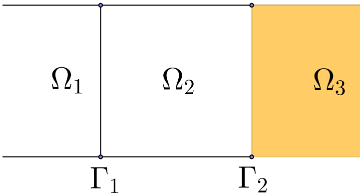
Given initial guess u^0, solve successively the following sub-problems:

(iii)

$$\begin{cases} \mathcal{L} v_3 &= f, & \text{in } \Omega_3, \\ \mathcal{B} v_3 &= g, & \text{on } \partial\Omega, \\ (\partial_{n_3} + \mathcal{S}_2^l)(v_3 - v_2) &= 0, & \text{on } \Gamma_2 \end{cases}$$

where $\mathcal{S}_1^l$ is the DtN operator on Γ_2 calculated with $\mathcal{L}$ in $\Omega_1 \cup \Omega_2$.

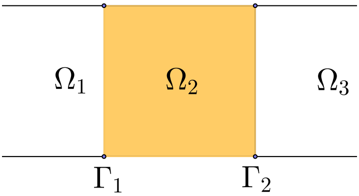
Schwarz Form of 'U' Solve at PDE Level



Algorithm 2.2 Schwarz form of the 'U' solve

Let $u_3 \leftarrow v_3$.

Schwarz Form of 'U' Solve at PDE Level



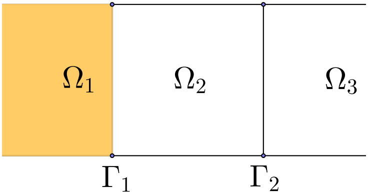
Algorithm 2.2 Schwarz form of the 'U' solve

Solve

(iv)

$$\begin{cases} \mathcal{L}u_2 = f, & \text{in } \Omega_2, \\ \mathcal{B}u_2 = g, & \partial\Omega, \\ (\partial_{n_3} + \mathcal{S}_2^I)(u_2 - v_1) = 0, & \text{on } \Gamma_1, \\ u_2 = u_3, & \text{on } \Gamma_2. \end{cases}$$

Schwarz Form of 'U' Solve at PDE Level

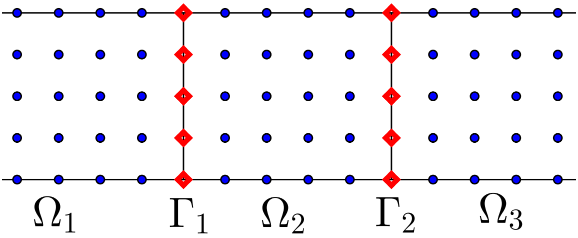


Algorithm 2.2 Schwarz form of the 'U' solve

Solve
(v)

$$\begin{cases} \mathcal{L}u_1 = f, & \text{in } \Omega_1, \\ \mathcal{B}u_1 = g, & \partial\Omega, \\ u_1 = u_2, & \text{on } \Gamma_1. \end{cases}$$

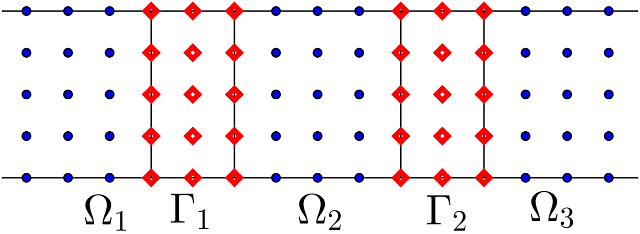
Partition of Nodes: Non-Overlapping



We partition the domain into subdomains $\widetilde{\Omega}_i$ with each subdomain further partitioned into boundary layers that are shared with other subdomains and non-shared interior, i.e.

$$\widetilde{\Omega}_i = \Gamma_{i-1} \cup \Omega_i \cup \Gamma_i.$$

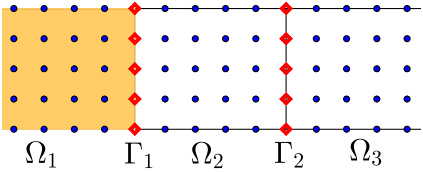
Partition of Nodes: Overlapping



We partition the domain into subdomains $\widetilde{\Omega}_i$ with each subdomain further partitioned into boundary layers that are shared with other subdomains and non-shared interior, i.e.

$$\widetilde{\Omega}_i = \Gamma_{i-1} \cup \Omega_i \cup \Gamma_i.$$

OSM of Symmetric Gauss-Seidel Type at Matrix Level



Algorithm 3.1 Optimal Schwarz method forward sweeping

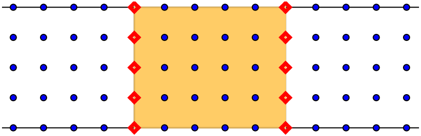
Given arbitrary initial guess $(\tilde{u}_j^{(0)},\ j = 1, 2, 3)$, at i-th iteration, we solve successively the following sub-problems:

(1)

$$\begin{pmatrix} A_{11} & A_{1\Gamma_1} \\ A_{\Gamma_1 1} & \underline{A}_{\Gamma_1\Gamma_1} \end{pmatrix} \begin{pmatrix} v_1^{(i)} \\ v_{\Gamma_1}^{(i)} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_{\Gamma_1} + (\underline{A}_{\Gamma_1\Gamma_1} - A_{\Gamma_1\Gamma_1}) u_{\Gamma_1}^{(i-1)} - A_{\Gamma_1 2} u_2^{(i-1)} \end{pmatrix},$$

where $\underline{A}_{\Gamma_1\Gamma_1}$ is arbitrary ensuring well-posedness.

OSM of Symmetric Gauss-Seidel Type at Matrix Level



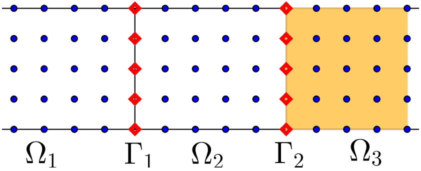
Algorithm 3.1 Optimal Schwarz method forward sweeping

Given arbitrary initial guess ($\tilde{u}_j^{(0)}$, $j = 1, 2, 3$), at i-th iteration, we solve successively the following sub-problems:
(2)

$$\begin{pmatrix} T_{\Gamma_1} & A_{\Gamma_1 2} & \\ A_{2\Gamma_1} & A_{22} & A_{2\Gamma_2} \\ & A_{\Gamma_2 2} & \underline{A}_{\Gamma_2\Gamma_2} \end{pmatrix} \begin{pmatrix} \bar{v}_{\Gamma_1}^{(i)} \\ v_2^{(i)} \\ v_{\Gamma_2}^{(i)} \end{pmatrix} = \begin{pmatrix} f_{\Gamma_1} + (T_{\Gamma_1} - A_{\Gamma_1\Gamma_1}) v_{\Gamma_1}^{(i)} - A_{\Gamma_1 1} v_1^{(i)} \\ f_2 \\ f_{\Gamma_2} + (\underline{A}_{\Gamma_2\Gamma_2} - A_{\Gamma_2\Gamma_2}) u_{\Gamma_2}^{(i-1)} - A_{\Gamma_2 3} u_3^{(i-1)} \end{pmatrix},$$

where $\underline{A}_{\Gamma_2\Gamma_2}$ is arbitrary ensuring well-posedness.

OSM of Symmetric Gauss-Seidel Type at Matrix Level



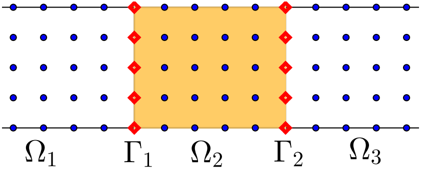
Algorithm 3.1 Optimal Schwarz method forward sweeping

Given arbitrary initial guess $(\tilde{u}_j^{(0)},\ j = 1, 2, 3)$, at i-th iteration, we solve successively the following sub-problems:

(3)

$$\begin{pmatrix} T_{\Gamma_2} & A_{\Gamma_2 3} \\ A_{3\Gamma_2} & A_{33} \end{pmatrix} \begin{pmatrix} u_{\Gamma_2}^{(i)} \\ u_3^{(i)} \end{pmatrix} = \begin{pmatrix} f_{\Gamma_2} + (T_{\Gamma_2} - A_{\Gamma_2\Gamma_2})v_{\Gamma_2}^{(i)} - A_{\Gamma_2 2}v_2^{(i)} \\ f_3 \end{pmatrix}.$$

OSM of Symmetric Gauss-Seidel Type at Matrix Level



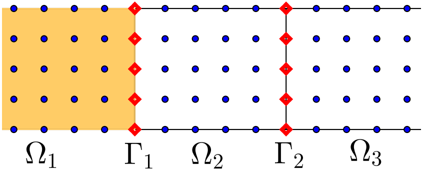
Algorithm 3.2 Optimal Schwarz method backward sweeping

Given arbitrary initial guess $(\tilde{u}_j^{(0)},\ j = 1, 2, 3)$, at i-th iteration, we solve successively the following sub-problems:
(4)

$$\begin{pmatrix} T_{\Gamma_1} & A_{\Gamma_1 2} & \\ A_{2\Gamma_1} & A_{22} & A_{2\Gamma_2} \\ & A_{\Gamma_2 2} & \underline{\underline{A}}_{\Gamma_2\Gamma_2} \end{pmatrix} \begin{pmatrix} u_{\Gamma_1}^{(i)} \\ u_2^{(i)} \\ \bar{u}_{\Gamma_2}^{(i)} \end{pmatrix} = \begin{pmatrix} f_{\Gamma_1} + (T_{\Gamma_1} - A_{\Gamma_1\Gamma_1}) v_{\Gamma_1}^{(i)} - A_{\Gamma_1 1} v_1^{(i)} \\ f_2 \\ f_{\Gamma_2} + (A_{\Gamma_2\Gamma_2} - \underline{\underline{A}}_{\Gamma_2\Gamma_2}) u_{\Gamma_2}^{(i)} - A_{\Gamma_2 3} u_3^{(i)} \end{pmatrix}.$$

where $\underline{\underline{A}}_{\Gamma_2\Gamma_2}$ is arbitrary ensuring well-posedness.

OSM of Symmetric Gauss-Seidel Type at Matrix Level



Algorithm 3.2 Optimal Schwarz method backward sweeping

Given arbitrary initial guess $(\tilde{u}_j^{(0)},\ j = 1, 2, 3)$, at i-th iteration, we solve successively the following sub-problems:

(5)

$$\begin{pmatrix} A_{11} & A_{1\Gamma_1} \\ A_{\Gamma_1 1} & \underline{\underline{A}}_{\Gamma_1\Gamma_1} \end{pmatrix} \begin{pmatrix} u_1^{(i)} \\ \bar{u}_{\Gamma_1}^{(i)} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_{\Gamma_1} + (\underline{\underline{A}}_{\Gamma_1\Gamma_1} - A_{\Gamma_1\Gamma_1}) u_{\Gamma_1}^{(i)} - A_{\Gamma_1 2} u_2^{(i)} \end{pmatrix},$$

where $\underline{\underline{A}}_{\Gamma_1\Gamma_1}$ is arbitrary ensuring well-posedness.

Convergence of the Optimal Schwarz Method

Theorem 4 (Gander, Z. '13)

If T_{Γ_i}, $i = 1, \ldots, J-1$ correspond to the algebraic DtN (i.e. Schur complements), then the optimal Schwarz method converges in one iteration consisting of one forward and one backward sweeping for arbitrary initial guess.

Theorem 5 (Gander, Z.'13)

If DtN transmission conditions are used on the left boundaries of subdomains, the optimal Schwarz method converges to the exact solution in one forward and one backward sweeping for arbitrary initial guess.

Proof.

Consider the error equation and make use of the definition of DtN to find vanishing boundary data on interfaces. □

Note that the theorems apply to both overlapping and non-overlapping methods.

Stolk's Rapidly Converging Domain Decomposition Method

Theorem 6 (Gander, Z. '13)

Stolk's rapidly converging DDM is a non-overlapping optimized Schwarz method of symmetric Gauss-Seidel type with zero initial guess and PML approximation of DtN transmission conditions on the left and right boundaries of subdomains under the assumption that the DtN is symmetric w.r.t. each interface.

Remark. As we have seen, the symmetry assumption is not necessary for one step convergence. This is important because the symmetry assumption does, for example, not hold for heterogeneous media.

Proof

For example, in the forward sweeping of the optimized Schwarz method

$$(\partial_{n_1} + \mathcal{S}_1^r) v_1^{(1)} = 0, \text{ on } \Gamma_1$$

is followed by

$$(\partial_{n_2} + \mathcal{S}_1^l) v_2^{(1)} = (-\partial_{n_1} + \mathcal{S}_1^l) v_1^{(1)} \text{ on } \Gamma_1.$$

Here, $\mathcal{S}_1^l(\mathcal{S}_1^r)$ is the DtN or its approximation calculated in the left(right) to Γ_1. Based on the assumption that $\mathcal{S}_1^r = \mathcal{S}_1^l$ and substituting the first equation to the second, we obtain

$$(\partial_{n_2} + \mathcal{S}_1^l) v_2^{(1)} = -2\partial_{n_1} v_1^{(1)},$$

which says $v_2^{(1)}$ has a non-zero Neumann jump $-2\partial_{n_1} v_1^{(1)}$ across Γ_1 between Ω_2 and the PML region. In other words, it has a Dirac source along Γ_1. This gives Stolk's DDM. □

Source Transfer Domain Decomposition Method

Theorem 7 (Chen, Gander, Z. '13)

The source transfer method is an overlapping optimized Schwarz method of symmetric Gauss-Seidel type with each overlap covering half subdomains that equips PML transmission conditions on the left and right boundaries in the forward sweeping and Dirichlet instead of PML on the right boundaries in the backward sweeping. Moreover, the source terms are consistently modified in the forward sweeping.

Many thanks to very helpful discussions with Zhiming Chen!

Settings for Numerical Examples

We solve on the unit square the Helmholtz equation with one point source. We use zero initial guess for GMRES with relative residual tolerance 10^{-6}.

- ω: wavenumber
- J: number of subdomains
- m: minimal number of pml grid layers on subdomain boundaries to reach the given number of iterations
- nx: number of grid layers including interior, overlap and pml, along x (partitioned dimension) in one subdomain
- PMLh: optimized Schwarz with PML and $2h$ overlap.

Is PML Scalable?

$\frac{\omega}{2\pi}$	$\frac{1}{10h}$	$\frac{J}{10}$	Sweep			Source			Stolk			PMLh		
			it	m	nx	it	m	nx	it	m	nx	it	m	nx
20	20	2	4	4	14	4	4	26, 22	3	3	16	4	2	16
20	40	4	4	11	21	4	7	32, 25	4	4	18	4	3	18
20	80	8	4	40	50	4	3	24, 21	4	8	26	4	7	26
40	40	4	6	5	15	5	4	26, 22	5	3	16	5	2	16
80	80	8	6	28	38	6	14	46, 32	5	4	18	6	3	18
160	160	16	11	47	57	11	34	86, 52	6	6	22	6	5	22

- ω: wavenumber, J: number of subdomains, it: GMRES iteration number
- m: minimal number of pml grid layers on subdomain boundaries to reach the given number of iterations
- nx: number of grid layers including interior, overlap and pml, along x (partitioned dimension) in one subdomain
- PMLh: optimized Schwarz with PML and $2h$ overlap.